Flows past sharp corners: the Brown-Michael Model

Abstract

We focus on two-dimensional separated flows past sharp corners. A vortex sheet is needed to render the velocity field finite at the corner, and the roll-up of the sheet is modelled by a vortex with tune-dependant circulation. Then the equation of motion for the starting vortex is given by from the Brown-Michael equation and solved either analytically or numerically in several cases. Finally the comparisons of the results obtained will be discussed.

Contents

1 Introduction

Two-dimensional flows past a sharp corner is an interesting topic in fluid dynamics and many studies have been made. In 1954, Brown and Michael studied the effect of leading edge separation and proposed a model that we use as a major equation in this paper. In 1992, Cortelezzi and Leonard investigated the problem that unsteady separated flows pass a semi-infinite plate. They used complex method to describe the two-dimensional plane. With an appropriate conformal mapping, a governing equation on the position of the starting vortex was found. Then in 1994, Cortelezzi gave an analytical method which can be used to solve for the exact solution to the governing equation. We will make a quick review of their work in section 2. To get begin, let us consider the problem of ideal flow past a corner (see figure 1).

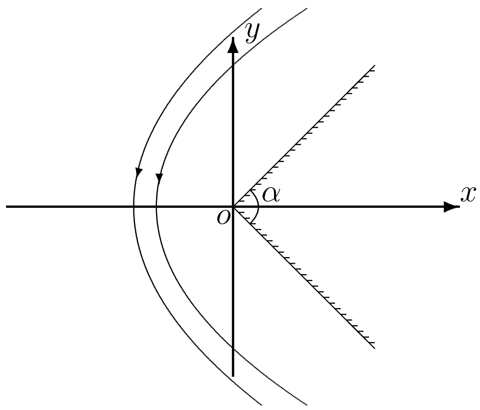

Figure 1: flows around at the wedge of α

A simple application of conformal mapping shows that the stream function $\psi = O(z^{\frac{\pi}{2\pi-\alpha}})$ near $z = 0$. For $\alpha < \pi$, $\frac{\pi}{2\pi-\alpha} \in (0,1)$. Therefore, the complex velocity is $w = u - \mathrm{i}v = \psi_y + \mathrm{i}\psi_x = O(z^{\frac{\pi}{2\pi-\alpha}-1})$. As $\frac{\pi}{2\pi-\alpha} - 1 \in (-1,0)$, the complex velocity is singular at the origin. However, physically there cannot be singularity in the velocity field and the body will shed vortices at the corner in the form of a vortex sheet so the velocity field is finite at the corner. Thus we introduce a continuous vortex sheet (see [4]) to describe the rolling-up of the shear layer. And the point vortex with time-dependent circulation (see [6]) will be used to model the vortex sheet in the physical plane (z-plane). In the case of flows past a semi-infinite plate, we perform a conformal mapping $z = -\mathrm{i}\zeta^2$. We derive the equation of motion for the starting vortex in the mapped plane from the Brown-Michael equation and then solve it both analytically and numerically. For the flows past a wedge with finite angle γ, we use a different mapping $z = -\mathrm{i}e^{\mathrm{i}\frac{\gamma}{2}}\zeta^{\frac{2\pi-\gamma}{\pi}}$. This time we can only investigate numerically. Comparisons of the solutions obtained in two cases will be discussed. Finally we look at the problem that flows pass two semi-infinite plates with a single gap. We follow the similar procedure as previously to get the problem solved numerically.

2 Separated flows past a semi-infinite plate

In this section we consider two-dimensional flow around a stationary semi-infinite plate. We assume that the shear layer, that separates from the plate tip and is convected away, is thin enough to justify a description by means of a vortex sheet. And following Brown-Michael dynamics we replace the vortex sheet with a point vortex. This model will not cause the complete loss of the vortex sheet. It is supposed to consist of a sheet of negligible circulation that connects the tip of the plate to a point vortex of variable strength, which represents the rolling up of the vortex sheet. The vortex and background flow satisfy an unsteady Kutta condition at the plate tip. The feeding vortex sheet can be considered as the branch cut because of the logarithmic singularity that represents the vortex.

2.1 Mathematical formulation

We would like to investigate the problem by using complex method, with the semi-infinite plate can be identified as the negative variable axis and tip at $z = 0$. Then we perform the following conformal mapping (see Figure 2),

$$z = -\mathrm{i}\zeta^2. \tag{1}$$

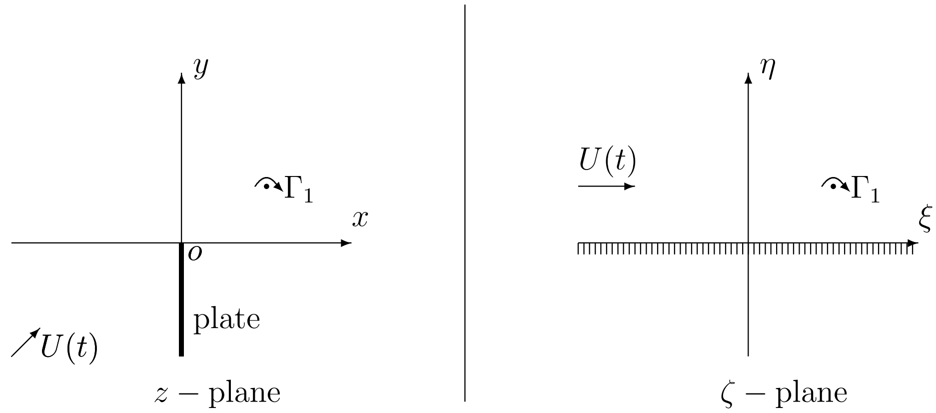

Figure 2: Flow past a semi-infinite plate in the physical z-plane and the mapped plane

The semi-infinite plate is then mapped onto the real axis in the ζ-plane with the domain laying above. i.e. $\mathrm{Im}(\zeta) > 0$. In addition, we denote U the free-stream velocity in the ζ-plane. We can express the complex potential F by simply superimposing the basic flows. It is well-known that the complex potential for a vortex is a logarithmic function with singularity at the vortex position. Therefore the complex velocity w in the ζ-plane can be written by differentiating the complex potential F with respect to ζ as below

$$w(\zeta,t) = \frac{\mathrm{d}F}{\mathrm{d}\zeta} = U(t) + \frac{\mathrm{i}\Gamma_1(t)}{2\pi}(\frac{1}{\zeta-\zeta_1(t)} - \frac{1}{\zeta-\bar{\zeta}_1(t)}) + \sum_{n=2}^{N} \frac{\mathrm{i}\Gamma_n}{2\pi}(\frac{1}{\zeta-\zeta_n(t)} - \frac{1}{\zeta-\bar{\zeta}_n(t)}). \quad (2)$$

where we used the method of images to satisfy the correct boundary condition at $\eta = 0$. Note that we have N vortices of strength Γ_n at position $\zeta = \zeta_n$ and its image of strength $-\Gamma_n$ (i.e. rotating in the anti-clockwise) at the complex conjugate position $\zeta = \bar{\zeta}_n$. Here Γ_1 depends on time so that the unsteady Kutta condition at $z = 0$ may be satisfied. By convention, the vortex of variable strength, i.e. the most recent vortex shed, is labeled with subscript 1, and then once a new vortex is shed all the others need to be renumbered. The potential flow in the physical plane presents a square root singularity nevertheless the flow in the ζ-plane is non-singular since the singularity at $z = 0$ has been removed by the conformal mapping. Let us consider the following chain rule

$$w_z = \frac{\mathrm{d}F}{\mathrm{d}z} = \frac{\mathrm{d}F}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}z} = w\frac{\mathrm{d}\zeta}{\mathrm{d}z},$$

where w_z is the complex velocity in the z-plane. As we said earlier, $\frac{\mathrm{d}\zeta}{\mathrm{d}z}$ is singular at the origin. Thus w has to be zero at the origin $\zeta = 0$ (i.e. the image of $z = 0$ under the map (1)) in the mapped plane in order to absorb the singularity of the velocity field w_z in the physical plane. We have

$$0 = U(t) + \frac{\mathrm{i}\Gamma_1(t)}{2\pi}(\frac{1}{-\zeta_1(t)} + \frac{1}{\bar{\zeta}_1(t)}) + \sum_{n=2}^{N}\frac{\mathrm{i}\Gamma_n}{2\pi}(\frac{1}{-\zeta_n(t)} + \frac{1}{\bar{\zeta}_n(t)}).$$

Solving for Γ_1,

$$\Gamma_1(t) = 2\pi\mathrm{i}(\frac{\zeta_1\bar{\zeta}_1}{\zeta_1 - \bar{\zeta}_1})[U + \sum_{n=2}^{N}\frac{\mathrm{i}\Gamma_n}{2\pi}(\frac{\zeta_n\bar{\zeta}_n}{\zeta_n - \bar{\zeta}_n})]. \tag{3}$$

To describe the motion mathematically, we need to use the Brown-Michael equation (see appendix A for the derivation), which is

$$\frac{\mathrm{d}\bar{z}_1}{\mathrm{d}t} + \frac{(\bar{z}_1 - \bar{z}_0)}{\Gamma_1}\frac{\mathrm{d}\Gamma_1}{\mathrm{d}t} = \lim_{z\to z_1}[\frac{\mathrm{d}}{\mathrm{d}z}(F - \frac{\mathrm{i}\Gamma_1}{2\pi}\log(z - z_1))], \tag{4}$$

$$\frac{\mathrm{d}\bar{z}_r}{\mathrm{d}t} = \lim_{z\to z_r}[\frac{\mathrm{d}}{\mathrm{d}z}(F - \frac{\mathrm{i}\Gamma_r}{2\pi}\log(z - z_r))]. \tag{5}$$

with initial conditions

$$z_1(t_s) = z_0,$$
$$z_r(t_s) = z_r,\ r = 2, ..., N.$$

and t_s is the shedding time. Equation (5) is the standard equation for a point vortex having constant circulation Γ_r, namely that it is adverted by the local velocity field (the term on the right-hand-side). Equation (4) is the so-called Brown-Michael equation (see appendix A for the derivation) first given by Brown and Michael, 1954, and is similar to (5) except that it contains an additional term involving the time derivative of the vortex circulation. In this model it holds until $\frac{d\Gamma_1}{dt} = 0$, i.e. the circulation stops increasing, where upon a new vortex is shed. Now we wish to solve the problem in the mapped plane. Thus we perform a change of variables which transforms z to ζ. With careful calculations (see appendix B.1), we get

$$
\begin{aligned}
&(2\mathrm{i}\bar{\zeta}_1 + \frac{\mathrm{i}\zeta_1\bar{\zeta}_1}{\zeta_1 - \bar{\zeta}_1})\frac{\mathrm{d}\bar{\zeta}_1}{\mathrm{d}t} - (\frac{\mathrm{i}\bar{\zeta}_1^3}{\zeta_1(\zeta_1 - \bar{\zeta}_1)})\frac{\mathrm{d}\zeta_1}{\mathrm{d}t} \\
&\quad = \frac{\mathrm{i}}{2\zeta_1}(U - \frac{\mathrm{i}\Gamma_1}{2\pi(\zeta_1 - \bar{\zeta}_1)} + \sum_{n=2}^{N} \frac{\mathrm{i}\Gamma_n}{2\pi} \frac{(\zeta_n - \bar{\zeta}_n)}{(\zeta_1 - \zeta_n)(\zeta_1 - \bar{\zeta}_n)} - \frac{\mathrm{i}\Gamma_1}{4\pi\zeta_1}) \\
&\quad\quad - \mathrm{i}\bar{\zeta}_1^2[\frac{\mathrm{d}U}{\mathrm{d}t} + \sum_{n=2}^{N} \frac{\mathrm{i}\Gamma_n}{2\pi}(\frac{1}{\zeta_n^2}\frac{\mathrm{d}\zeta_n}{\mathrm{d}t} - \frac{\mathrm{d}\bar{\zeta}_n}{\mathrm{d}t})](U + \sum_{n=2}^{N} \frac{\mathrm{i}\Gamma_n}{2\pi}\frac{\zeta_n - \bar{\zeta}_n}{\zeta_n\bar{\zeta}_n})^{-1}. \quad (6)
\end{aligned}
$$

$$
\frac{\mathrm{d}\bar{\zeta}_r}{\mathrm{d}t} = \frac{1}{4\zeta_r\bar{\zeta}_r}(U - \frac{\mathrm{i}\Gamma_r}{2\pi(\zeta_r - \bar{\zeta}_r)} + \sum_{n\neq r} \frac{\mathrm{i}\Gamma_n}{2\pi}\frac{(\zeta_n - \bar{\zeta}_n)}{(\zeta_1 - \zeta_n)(\zeta_1 - \bar{\zeta}_n)} - \frac{\mathrm{i}\Gamma_r}{4\pi\zeta_r}). \quad (7)
$$

with the initial conditions

$$
\begin{aligned}
&\zeta_1(t_s) = 0, \\
&\zeta_r(t_s) = \zeta_{r_s},\ r = 2, ..., N.
\end{aligned}
$$

The above equations were first given by Cortelezzi and Lenoard, 1992 (see [1]). To simplify the problem, we will only consider the starting vortex (i.e. the vortex with subscript 1). The reduced equations are

$$
\begin{aligned}
&(2\mathrm{i}\bar{\zeta}_1 + \frac{\mathrm{i}\zeta_1\bar{\zeta}_1}{\zeta_1 - \bar{\zeta}_1})\frac{\mathrm{d}\bar{\zeta}_1}{\mathrm{d}t} - (\frac{\mathrm{i}\bar{\zeta}_1^3}{\zeta_1(\zeta_1 - \bar{\zeta}_1)})\frac{\mathrm{d}\zeta_1}{\mathrm{d}t} \\
&\quad\quad = \frac{\mathrm{i}}{2\zeta_1}(U - \frac{\mathrm{i}\Gamma_1}{2\pi(\zeta_1 - \bar{\zeta}_1)} - \frac{\mathrm{i}\Gamma_1}{4\pi\zeta_1}) - \frac{\mathrm{i}\bar{\zeta}_1^2}{U}\frac{\mathrm{d}U}{\mathrm{d}t}. \quad (8)
\end{aligned}
$$

with initial condition

$$
\zeta_1(0) = 0. \quad (9)
$$

From (3), we know that

$$
\Gamma_1(t) = 2\pi\mathrm{i}(\frac{\zeta_1\bar{\zeta}_1}{\zeta_1 - \bar{\zeta}_1})U. \quad (10)
$$

Then we focus on investigating equation (8) in the remaining of this chapter.

2.2 Exact solution

Without losing generality, we assume that the free-stream velocity is always positive. Now we would like to seek exact solutions (first given by Cortelezzi in 1994, see [2]). First rewrite it in polar coordinates $\zeta = \rho e^{\mathrm{i}(\frac{\pi}{2}-\theta)} = \mathrm{i}\rho e^{-\mathrm{i}\theta}$.

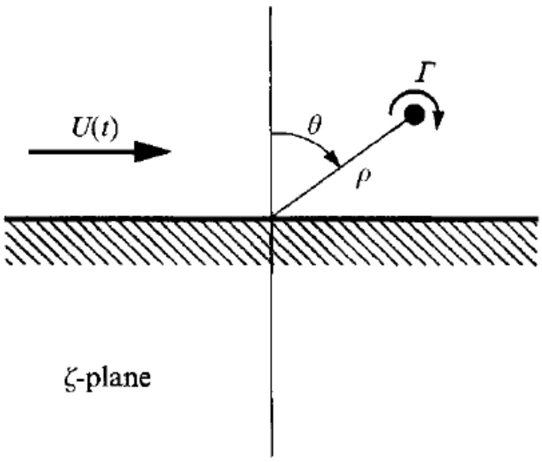

Figure: Polar Coordinates.

After some simplifications (see details in appendix C), we have

$$\frac{\mathrm{d}\rho_1}{\mathrm{d}t} = \frac{U\sin(\theta_1)}{12\rho_1^2} - \frac{\rho_1}{3U}\frac{\mathrm{d}U}{\mathrm{d}t}, \tag{11}$$

$$\frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \frac{U\cos(2\theta_1)}{8\rho_1^3\cos(\theta_1)}. \tag{12}$$

with the initial conditions

$$\rho_1(0) = 0, \tag{13}$$

$$\theta_1(0) = \theta_0,\ \theta_0 \in [-\frac{\pi}{2}, \frac{\pi}{2}]. \tag{14}$$

Let us define three new variables (first proposed by McLaughlin *et al*, 1986, see [3])

$$\alpha = U\rho_1^3,$$
$$\beta = \sin(\theta_1),$$
$$\tilde{t} = \int_0^t U^2(\tau)\mathrm{d}\tau.$$

Compute the derivative of α and β with respect to $\tilde{t}$,

$$\begin{aligned}\frac{\mathrm{d}\alpha}{\mathrm{d}\tilde{t}} &= \frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tilde{t}} = (\frac{\mathrm{d}U}{\mathrm{d}t}\rho_1^3 + 3U\rho_1^2\frac{\mathrm{d}\rho_1}{\mathrm{d}t})\frac{1}{U^2} \\ &= [\frac{\mathrm{d}U}{\mathrm{d}t}\rho_1^3 + 3U\rho_1^2(\frac{U\sin(\theta_1)}{12\rho_1^2} - \frac{\rho_1}{3U}\frac{\mathrm{d}U}{\mathrm{d}t})]\frac{1}{U^2} \\ &= \frac{\sin(\theta_1)}{4} = \frac{\beta}{4}.\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}\beta}{\mathrm{d}\tilde{t}} &= \cos(\theta_1)\frac{\mathrm{d}\theta_1}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tilde{t}} = \frac{U\cos(2\theta_1)}{8\rho_1^3}\frac{1}{U^2} \\ &= \frac{1-2\sin^2(\theta_1)}{8\alpha} = \frac{1-2\beta^2}{8\alpha}.\end{aligned}$$

with initial conditions $\alpha(0) = 0, \beta(0) = \beta_0,\ \beta_0 \in (-1, 1)$.
Note that

$$\begin{aligned}\frac{\mathrm{d}^2\alpha^2}{\mathrm{d}\tilde{t}^2} &= \frac{\mathrm{d}}{\mathrm{d}\tilde{t}}(2\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}t}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}(\alpha\beta) \\ &= \frac{1}{2}(\frac{\mathrm{d}\alpha}{\mathrm{d}\tilde{t}}\beta + \frac{\mathrm{d}\beta}{\mathrm{d}\tilde{t}}\alpha) = \frac{1}{2}(\frac{\beta^2}{4} + \frac{1-2\beta^2}{8}).\end{aligned}$$

So

$$\frac{\mathrm{d}^2\alpha^2}{\mathrm{d}\tilde{t}^2} = \frac{1}{16}. \tag{15}$$

We can easily obtain the solution by integrating (15) twice with respect to $\tilde{t}$,

$$\frac{\mathrm{d}(\alpha^2)}{\mathrm{d}\tilde{t}} = 2\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}\tilde{t}} = \frac{\tilde{t}}{16} + a_1, \tag{16}$$

$$\alpha^2 = \frac{\tilde{t}^2}{32} + a_1\tilde{t} + a_0. \tag{17}$$

where a_1, a_0 are constants to be determined by the initial conditions. We know $a_0 = 0$ from (17) and $a_1 = 0$ from (16) since $\alpha(0) = 0$. Therefore, $\alpha^2 = \frac{\tilde{t}^2}{32}$. The exact solution has the following form:

$$\alpha = \pm\frac{\tilde{t}}{4\sqrt{2}}, \tag{18}$$

$$\beta = \pm\frac{\sqrt{2}}{2}. \tag{19}$$

We choose the positive sign because we assumed initially that the free-stream velocity is positive at any time. Then we can solve for ρ_1 and θ_1,

$$\rho_1 = (\frac{\alpha}{U})^{\frac{1}{3}} = [\frac{1}{4\sqrt{2}U}\int_0^t U^2(\tau)\mathrm{d}\tau]^{\frac{1}{3}}, \tag{20}$$

$$\theta_1 = \frac{\pi}{4}. \tag{21}$$

Finally we invert the solution into the physical plane,

$$z_1 = -\mathrm{i}(\rho_1 e^{\mathrm{i}(\frac{\pi}{2}-\theta_1)})^2 = \rho_1^2 = [\frac{1}{4\sqrt{2}U}\int_0^t U^2(\tau)\mathrm{d}\tau]^{\frac{2}{3}}, \tag{22}$$

From (10), we have

$$\Gamma_1 = 2\pi\mathrm{i}U\frac{\rho_1^2}{2\rho_1\mathrm{i}\sin(\theta_1)} = \pi U\frac{\rho_1}{\beta} \tag{23}$$

$$= \pi(\frac{U^2}{2}\int_0^t U^2(\tau)\mathrm{d}\tau)^{\frac{1}{3}}. \tag{24}$$

Differentiate Γ_1 with respect to t,

$$\frac{\mathrm{d}\Gamma_1}{\mathrm{d}t} = \frac{\pi}{3}(\frac{dU}{dt}\int_0^t U^2(\tau)\mathrm{d}\tau + \frac{U^3}{2})(\frac{U^2}{2}\int_0^t U^2(\tau)\mathrm{d}\tau)^{-\frac{2}{3}}. \tag{25}$$

Note if U is constant and $\frac{d\Gamma_1}{dt} \neq 0$ for all time i.e. the circulation Γ_1 continues to increase and there is no need for another vortex to be shed. In fact $\Gamma_1 = \pi(\frac{U^4 t}{2})^{\frac{1}{3}}$ from (24) if U is constant. More generally at $t = t_s$, $\frac{\mathrm{d}\Gamma_1}{\mathrm{d}t} = 0$, which implies that

$$\frac{dU}{dt}\int_0^t U^2(\tau)\mathrm{d}\tau + \frac{U^3}{2} = 0.$$

Therefore,

$$[\frac{dU}{dt}]_{t=t_s} = -\frac{U^3}{2}(\int_0^{t_s} U^2(\tau)\mathrm{d}\tau)^{-1}. \tag{26}$$

Mathematically we have solved the problem and obtained the exact solution. In reality, the physical interpretation may depend on the particular situation.

2.3 Numerical solution and simulation

In this section, we seek to solve the problem numerically and then simulate the numerical solution which can be compared to the exact one we obtained in the previous section.

2.3.1 Case of steady flow

To get started, let us review equation (8) in the case where U is steady.

$$(2\mathrm{i}\bar{\zeta}_1+\frac{\mathrm{i}\zeta_1\bar{\zeta}_1}{\zeta_1-\bar{\zeta}_1})\frac{\mathrm{d}\bar{\zeta}_1}{\mathrm{d}t}-(\frac{\mathrm{i}\bar{\zeta}_1^3}{\zeta_1(\zeta_1-\bar{\zeta}_1)})\frac{\mathrm{d}\zeta_1}{\mathrm{d}t}=\frac{\mathrm{i}}{2\zeta_1}(U-\frac{\mathrm{i}\Gamma_1}{2\pi(\zeta_1-\bar{\zeta}_1)}-\frac{\mathrm{i}\Gamma_1}{4\pi\zeta_1}). \quad (27)$$

Writing $\zeta_1=\xi+\mathrm{i}\eta$, (27) can be converted into (see appendix D)

$$\dot{\xi}=\frac{U}{24}\frac{3\eta^2-\xi^2}{(\eta^2+\xi^2)^2}, \quad (28)$$

$$\dot{\eta}=-\frac{U}{24}\frac{\xi(\eta^2-3\xi^2)}{\eta(\eta^2+\xi^2)^2}. \quad (29)$$

Equations (28) and (29) are a non-linear dynamical system. We use two-dimensional Runge-Kutta 4 to solve and simulate the solution via an ODE software called "pplane" in Matlab.

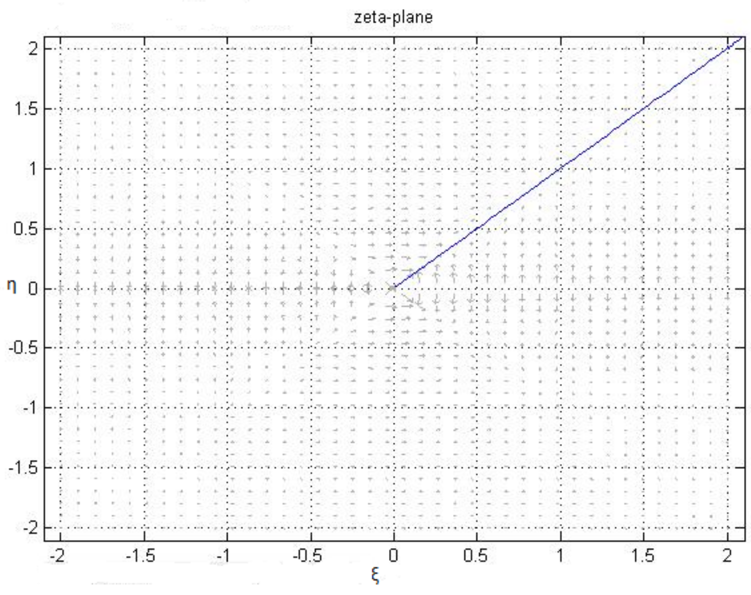

Figure 3: Trajectory of the vortex in the case of steady flows
in the ζ-plane

The curve represents the numerical solution to the dynamical system (28) & (29) whose initial point is chosen to be the origin. The graph shows the vortex always follows the line $\xi = \eta$ in the ζ-plane. i.e. The angle θ_1 is constantly $\frac{\pi}{4}$, as expected from the exact solution (21). We draw the graphs of $\xi(t)$ (see Figure 4) and $x(t)$ (see Figure 5) to see how these two functions behave in term of t.

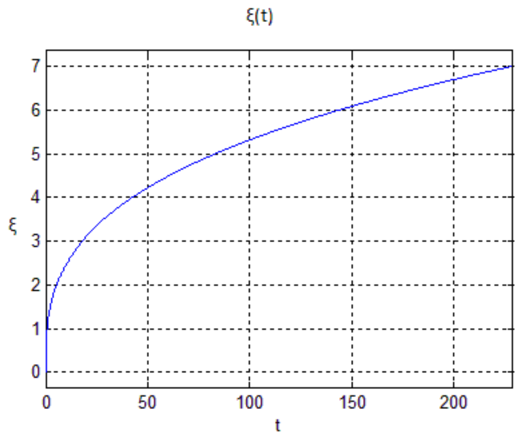

Figure 4: The graph of $\xi(t)$.

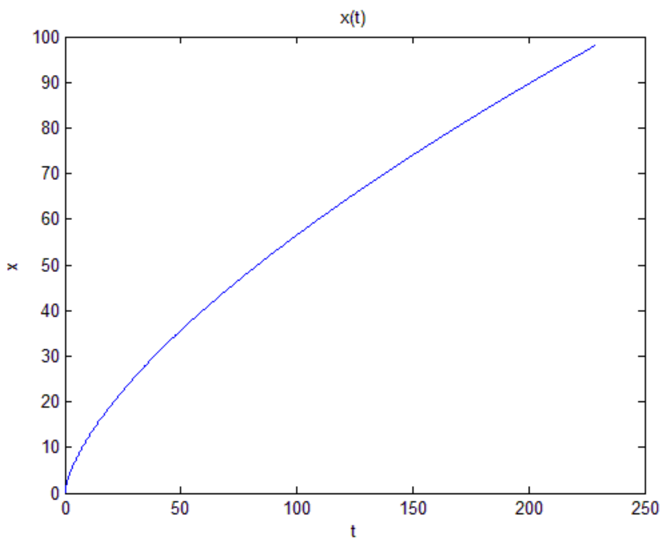

Figure 5: The graph of $x(t)$.

(20) reveals $\rho_1 = O(t^{\frac{1}{3}})$ in the case where U is constant. Figure 5 shows the same information. Testing some values from the points on the curve from Figure 6, we obtain that $x(t) = O(t^{\frac{2}{3}})$. It perfectly coincides with the exact solution (22). Using the conformal mapping (1) we are able to find a relationship between (x, y) and $(\xi.\eta)$. Note that

$$x + \mathrm{i}y = z = -\mathrm{i}\zeta^2 = 2\xi\eta + \mathrm{i}(\eta^2 - \xi^2). \tag{30}$$

Therefore

$$x = 2\xi\eta, \tag{31}$$
$$y = \eta^2 - \xi^2. \tag{32}$$

Hence $y = 0$ for any t and $x = 2\xi^2 = O(t^{\frac{2}{3}})$ on the trajectory of the vortex. Then the trajectory of the vortex in the physical z-plane is as the following,

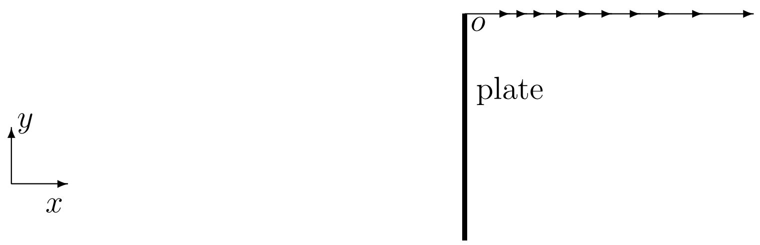

Figure 6: Trajectory of the vortex in the physical plane.

Note that in the case where U is constant, $\frac{\mathrm{d}\Gamma_1}{\mathrm{d}t} > 0$ for any $t > 0$. Therefore Γ_1 keeps increasing all the time and there is no other vortex being shed.

2.3.2 Case of unsteady flow

In this subsection we do the similar job as what we did previously. And we will obtain

$$\dot{\xi} = \frac{U}{24}\frac{3\eta^2 - \xi^2}{(\eta^2 + \xi^2)^2} - \frac{\xi\dot{U}}{3U} \tag{33}$$

$$\dot{\eta} = -\frac{U}{24}\frac{\xi(\eta^2 - 3\xi^2)}{\eta(\eta^2 + \xi^2)^2} - \frac{\eta\dot{U}}{3U} \tag{34}$$

Take $U(t) = 1 - \exp(-\omega t)$, where ω is a parameter. This represents a flow which is "slowly turned on"from $U = 0$ at $t = 0$ to $U = 1$ at $t \to \infty$. We put $\omega = 1$ and solve for the numerical solution. Plot the graphs of the numerical and exact solutions using Mathematica,

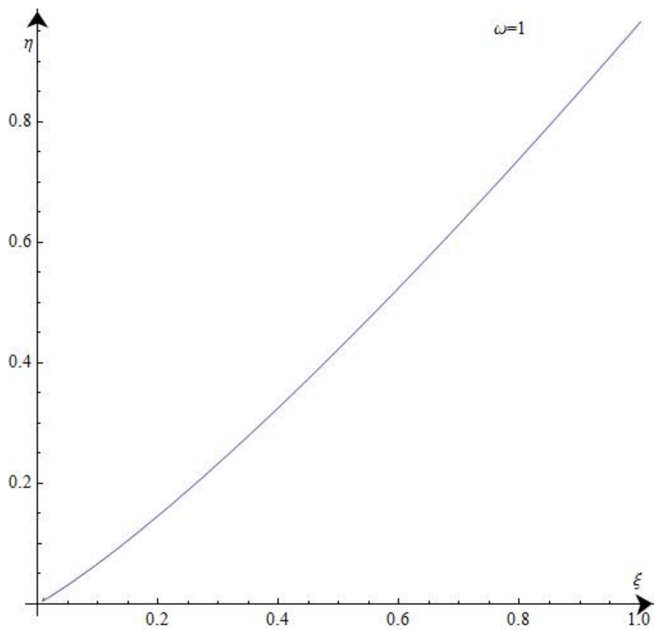

Figure 7: The numerical solution to the problem of unsteady flows past a semi-infinite plate.

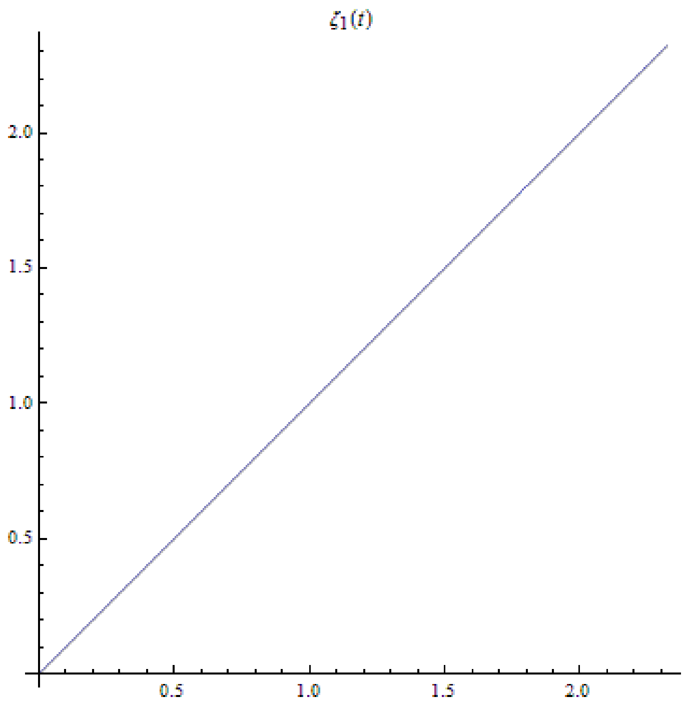

Figure 8: The exact solution to the problem of unsteady flows past a semi-infinite plate.

The curve of the numerical solution is nearly a straight line with slope 1. Meanwhile we know θ_1 always equals $\frac{\pi}{4}$ from the exact solution and so that the curve is also a line with slope 1. Therefore the numerical scheme is accurate and effective.

3 Separated flows past a finite angle wedge

Previously we studied separated flows past a semi-infinite plate. In this section we work on the wedge with a finite angle γ.

3.1 Mathematical formulation

The equation describing the motion remains valid. This time we need to define a new conformal mapping (see appendix E.1 for the derivation),

$$z = -\mathrm{i}e^{\mathrm{i}\frac{\gamma}{2}}\zeta^{\frac{2\pi-\gamma}{\pi}}. \tag{35}$$

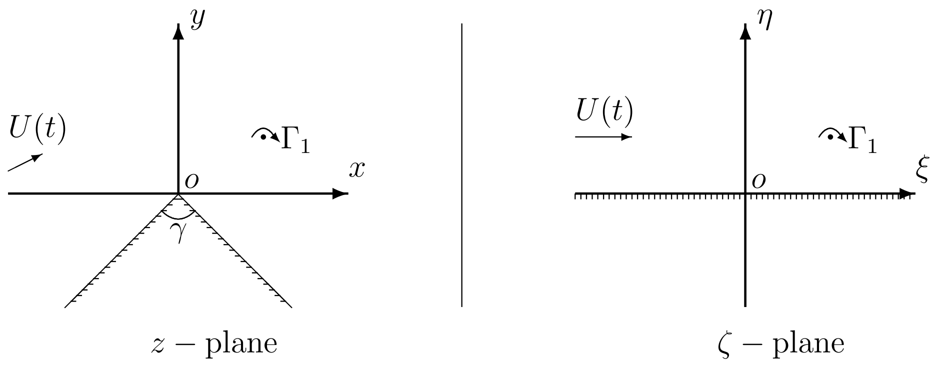

Figure 9: Flows past a wedge with angle γ in the physical z-plane and in the mapped ζ-plane.

It can be seen that the previous problem is a particular case of this one with $\gamma = 0$. If we replace γ with 0 in (35), it will become $z = -\mathrm{i}\zeta^2$ which maps a upper-half-plane semi-infinite plate onto the real axis. This result totally agrees with what we got in section 2.1. We let $k = \frac{2\pi-\gamma}{\pi}$. Note that k takes value from 1 to 2. As we did previously in section 2.1, the Brown-Michael equation can be written in the mapped plane by using the conformal mapping (35) as below,

$$\frac{\mathrm{i}\dot{\bar{\zeta}}_1}{\bar{\zeta}_1}(k + \frac{\zeta_1}{\zeta_1 - \bar{\zeta}_1}) - \frac{\mathrm{i}\dot{\zeta}_1}{\zeta_1}\frac{\bar{\zeta}_1}{(\zeta_1 - \bar{\zeta}_1)} = \frac{\mathrm{i}}{k\bar{\zeta}_1(\zeta_1\bar{\zeta}_1)^{k-1}}(U - \frac{\mathrm{i}\Gamma_1}{2\pi(\zeta_1 - \bar{\zeta}_1)} - \frac{\mathrm{i}\Gamma_1}{4\pi\zeta_1}) - \frac{\mathrm{i}\dot{U}}{U}. \quad (36)$$

with the initial conditions $\zeta_1(0) = 0$. As we did earlier, we want to switch to polar coordinates. $\zeta_1 = \rho_1 e^{\mathrm{i}(\frac{\pi}{2}-\theta_1)} = \mathrm{i}\rho_1 e^{-\mathrm{i}\theta_1}$. The L.H.S. of (36) becomes

$$\text{L.H.S.} = (k+1)\frac{\dot{\rho}_1}{\rho_1} + \dot{\theta}_1 \tan\theta_1 + \mathrm{i}k\dot{\theta}_1.$$

And the R.H.S. is

$$\text{R.H.S.} = \frac{U}{k\rho_1^{2k-1}}(\frac{\sin\theta_1(4\cos^2\theta_1 - 1)}{4\cos^2\theta_1} + \mathrm{i}\frac{\cos 2\theta_1}{2\cos\theta_1}) - \frac{\dot{U}}{U}.$$

After simplifications, we get two equations

$$\dot{\theta}_1 = \frac{U \cos 2\theta_1}{2k^2 \rho_1^{2k-1} \cos\theta_1}. \tag{37}$$

$$\dot{\rho}_1 = \frac{U}{(k+1)k^2\rho_1^{2k-2}}[(k-1)\sin\theta_1 + \frac{2-k}{4}\frac{\sin\theta_1}{\cos^2\theta_1}] - \frac{\rho_1}{k+1}\frac{\dot{U}}{U}. \tag{38}$$

with initial conditions

$$\rho_1(0) = 0, \tag{39}$$

$$\theta_1(0) = \theta_0 \in [-\frac{\pi}{2}, \frac{\pi}{2}]. \tag{40}$$

3.2 Analysis on the dynamical system

We assume that flows come from the left to the right (i.e. $U(t)$ is always positive) so that θ_1 is positive. Therefore, $\theta_0 \in [0, \frac{\pi}{2}]$. We are not able to find general solutions to the problem with an arbitrary angle. However we can analyse the dynamical system to gain more information. From (37), we know that $\theta_1 = \frac{\pi}{4}$ is a fixed point of this differential equation. In addition, $\dot{\theta}_1$ is always positive when $\theta_1 < \frac{\pi}{4}$. Thus it will not stop increasing until it reaches $\frac{\pi}{4}$. Similarly for the other side, $\dot{\theta}_1$ is always negative when $\theta_1 > \frac{\pi}{4}$. Hence $\theta_1 = \frac{\pi}{4}$ is a stable fixed point and is attractive in the first quadrant $\mathbb{R}_{++}$. Note that $\dot{\theta}_1$ is singular initially since $\rho_1(0) = 0$. The only way to remove the singularity is that $\theta_0 = \frac{\pi}{4}$. It implies that $\theta_1 = \frac{\pi}{4}$ for any t.

Now we assume that U is constant, then (38) becomes

$$\dot{\rho}_1 = \frac{\sqrt{2}U}{4(k+1)k}\frac{1}{\rho_1^{2k-2}}. \tag{41}$$

This equation can be easily solved subject to the initial condition and we get

$$\rho_1 = (\lambda t)^{\frac{1}{2k-1}}. \tag{42}$$

where

$$\lambda = \frac{\sqrt{2}U(2k-1)}{4(k+1)k}. \tag{43}$$

Now we look back in the physical plane. The conformal mapping is $z = -\mathrm{i}e^{\mathrm{i}\frac{\gamma}{2}}\zeta^k$, which gives

$$z = -\mathrm{i}e^{\mathrm{i}\frac{\gamma}{2}}\rho_1^k e^{\frac{\mathrm{i}k\pi}{4}} = \rho_1^k e^{\mathrm{i}(\frac{\gamma}{2}+\frac{k\pi}{4}-\frac{\pi}{2})} = \rho_1^k e^{\mathrm{i}\frac{\gamma}{4}}. \tag{44}$$

Therefore

$$z = (\lambda t)^{\frac{k}{2k-1}} e^{\mathrm{i}\frac{\gamma}{4}}. \tag{45}$$

Recall that $\gamma = 0$ ($k = 2$) represents the case of semi-infinite plate. Here we obtain the same result as (20), (21) and (22). In the z-plane, the trajectory of the vortex is shown as below

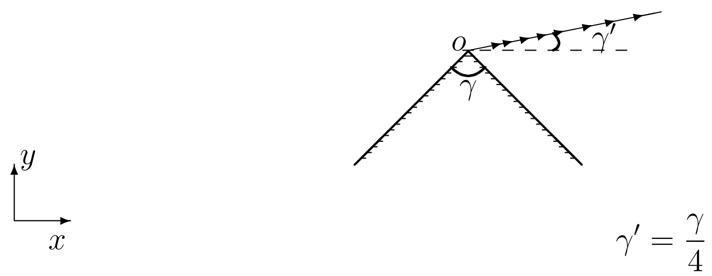

Figure 10: Trajectory of the vortex in the physical z-plane

Theoretically, we solve the problem. However in the reality the result could be slightly different from the exact solution. Some flow visualization experiments on this topic (see [5]) show that the position of the starting vortex is random initially and the trajectory is not a straight line within a very short time interval. It is due to the initial conditions. As the starting vortex cannot be perfectly generated at the tip of the wedge, the initial point is randomly chosen very close to the origin. Although it starts differently from what we have in the exact solution, θ_1 converges rapidly to $\frac{\pi}{4}$ due to the result we got earlier in this section. Therefore there is no difference in solution over a long time interval. We check the stability of the fixed point $\theta_1 = \frac{\pi}{4}$ numerically in the following section.

3.3 Numerical solution to the problem

In this section, our objective is to find numerical solutions to the governing equation (36). We rewrite it by letting $\zeta = \xi + \mathrm{i}\eta$. Then we can get two equations by collection the terms from the real part and the imaginary part respectively as following

$$\dot{\xi} = \frac{U}{(\xi^2+\eta^2)^k}[\frac{2-k}{4k^2(1+k)}\frac{\xi^4}{\eta^2} + \frac{1}{2k^2}\eta^2 + \frac{k-4}{4k^2(1+k)}\xi^2] - \frac{\dot{U}\xi}{U(1+k)}, \quad (46)$$

$$\dot{\eta} = \frac{U\xi}{\eta(\xi^2+\eta^2)^k}[\frac{k-4}{4k^2(1+k)}\eta^2 + \frac{k+4}{4k^2(1+k)}\xi^2] - \frac{\dot{U}\eta}{U(1+k)}. \quad (47)$$

Assuming that flows are steady (i.e. $\dot{U} = 0$), we solve the system by using numerical methods and simulate the solutions via Matlab,

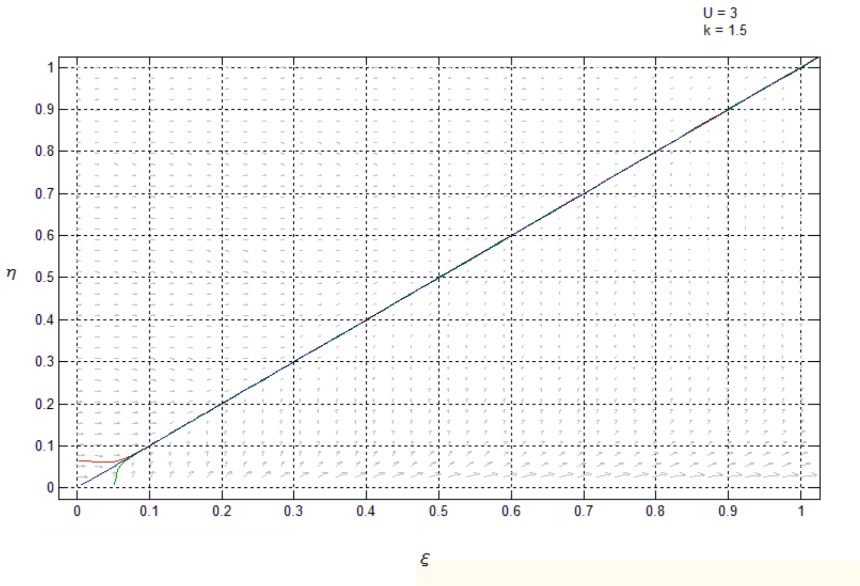

Figure 11: Trajectories of the vortex in the mapped plane with several different initial conditions when $\gamma = \frac{\pi}{2}$.

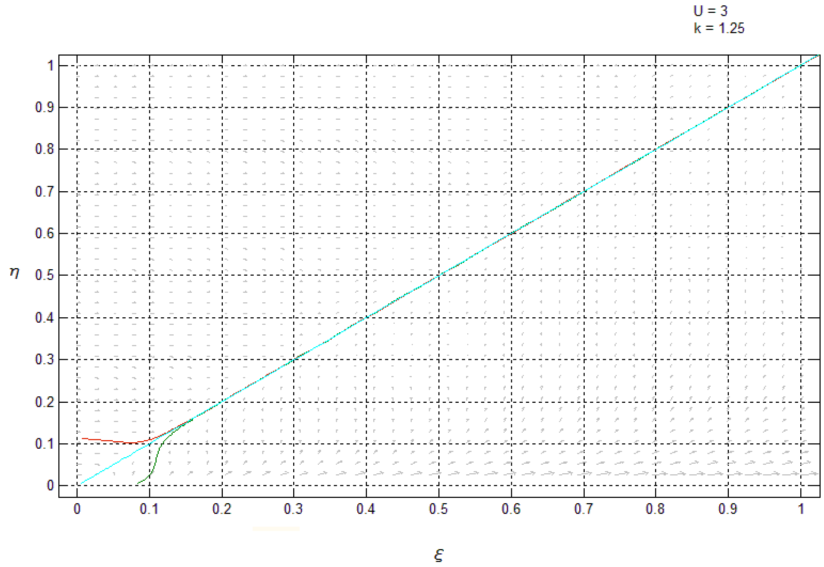

Figure 12: Trajectories of the vortex in the mapped plane with several different initial conditions when $\gamma = \frac{3\pi}{4}$.

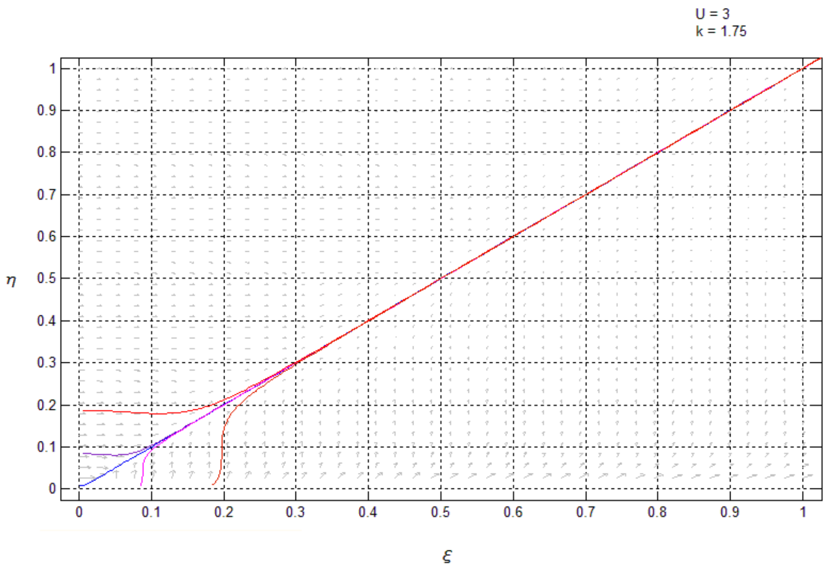

Figure 13: Trajectories of the vortex in the mapped plane with several different initial conditions when $\gamma = \frac{\pi}{4}$.

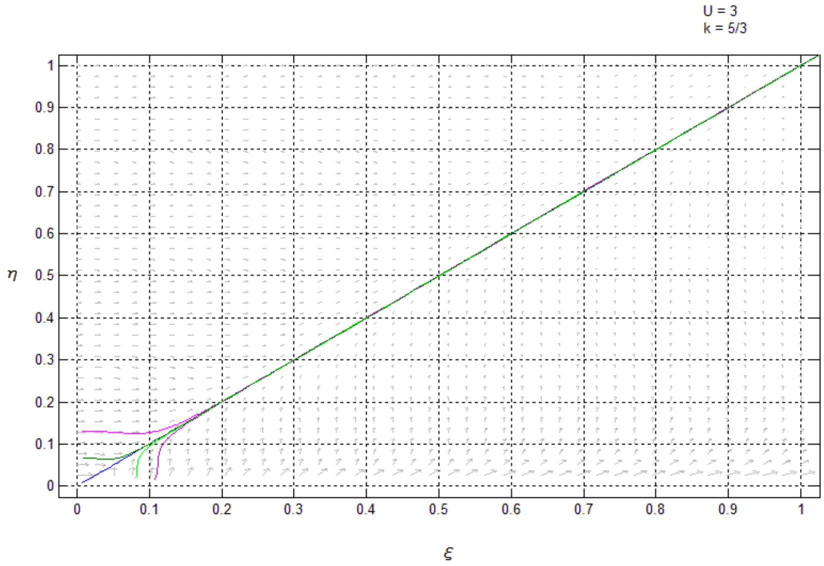

Figure 14: Trajectories of the vortex in the mapped plane with several different initial conditions when $\gamma = \frac{\pi}{6}$.

As can be seen from the graphs above, $\theta_1 = \frac{\pi}{4}$ is an absorbing steady state.

4 Separated flows past two semi-infinite plates

Now we consider the problem that flows pass two semi-infinite plates with a single gap. As earlier, we start with defining a conformal mapping which maps the region in the physical z-plane to the upper-half plane in the ζ-plane (see Figure 15) as the following.

$$\zeta = z + \sqrt{z^2 - 1}. \tag{48}$$

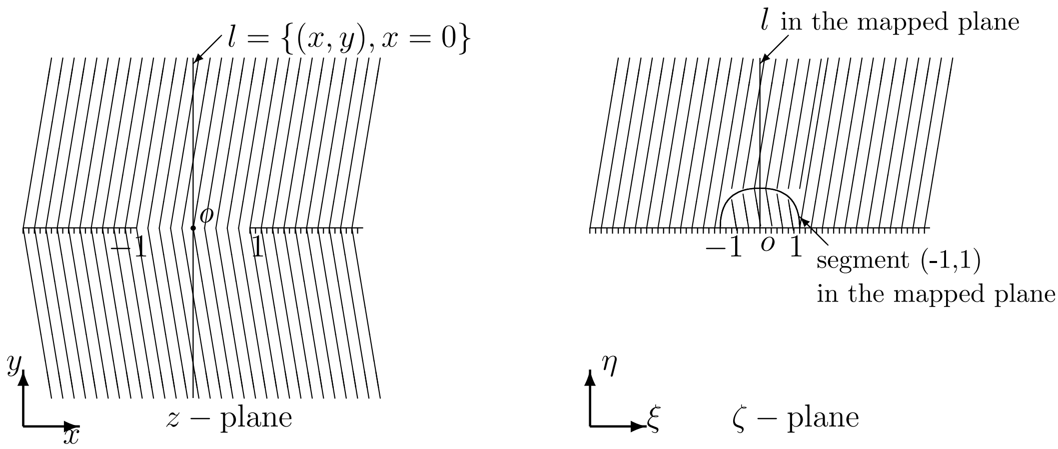

Figure 15: The physical plane and the mapped plane.

The complex potential of the background flow is $F = \pm\frac{Q}{2\pi}\ln\zeta = \pm\frac{Q}{2\pi}\ln(z + \sqrt{z^2-1})$ and Q is constant. As we all know, the streamfunction $\psi = \mathrm{Im}F$. Following from that, we can plot the streamfunction to show the streamlines (see Figure 16) and the graph of flows past two semi-infinite plates with a single gap (see Figure 17).

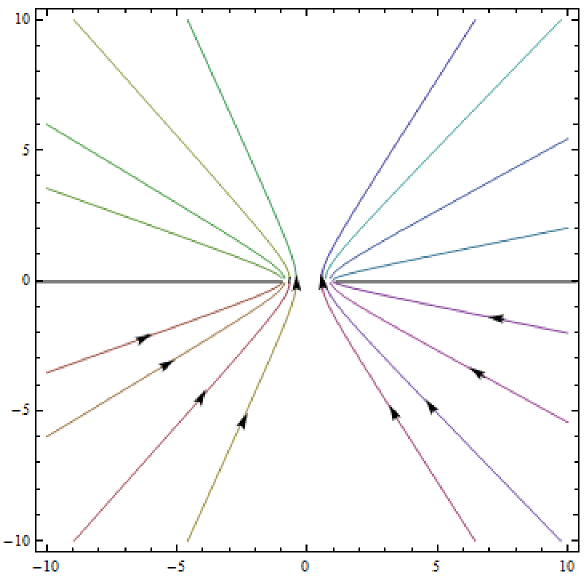

Figure 16: The streamlines in the z-plane.

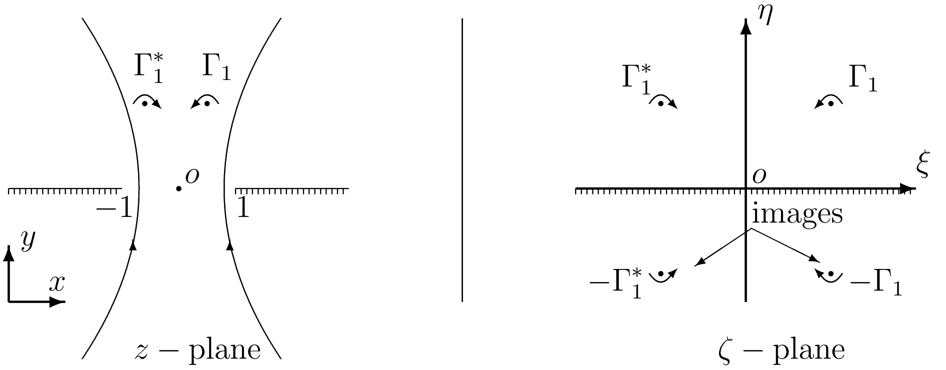

Figure 17: Flows past two semi-infinite plates with a single gap in the physical z-plane and in the mapped ζ-plane.

Note that (48) maps the two semi-infinite plates with a single gap to the real-axis (i.e. ξ-axis). The upper-half (lower-half respectively) plane is mapped onto the exterior (interior) of the half unit circle in the mapped plane. In addition, the segment $\{z = x \in \mathbb{R}, z \in (-1, 1)\}$ is mapped onto the half unit circle in the ζ-plane.

4.1 Mathematical formulation

Now we rewrite the mapping as

$$z = \frac{\zeta^2 + 1}{2\zeta}. \tag{49}$$

By symmetry, $\zeta_1^* = -\bar{\zeta}_1$ and $\Gamma_1^*(t) = -\Gamma_1(t)$.

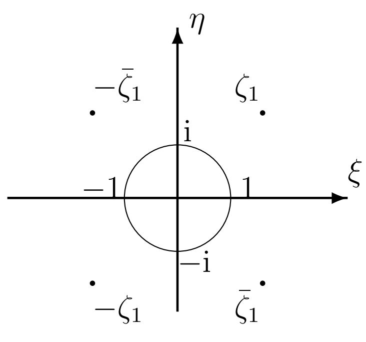

Figure 18: The positions of the vortices in the mapped plane

The complex velocity in the mapped plane is

$$w = \frac{\mathrm{d}F}{\mathrm{d}\zeta} = \frac{Q}{2\pi\zeta} - \frac{\mathrm{i}\Gamma_1(t)}{2\pi}(\frac{1}{\zeta-\zeta_1} - \frac{1}{\zeta-\bar{\zeta_1}}) + \frac{\mathrm{i}\Gamma_1(t)}{2\pi}(\frac{1}{\zeta+\bar{\zeta_1}} - \frac{1}{\zeta+\zeta_1}). \quad (50)$$

where F is the complex potential and $\frac{Q}{2\pi\zeta}$ is the background flow velocity. w has to be zero at $\zeta = \pm 1$ (corresponds to $z = \pm 1$) to remove the singularity. We obtain the following relationship,

$$\Gamma_1(t) = \frac{Q\mathrm{i}}{2}\frac{(1-\zeta_1^2)(1-\bar{\zeta_1^2})}{(\bar{\zeta_1^2}-\zeta_1^2)}. \quad (51)$$

Then we write the Brown-Michael equation

$$\frac{\mathrm{d}\bar{z_1}}{\mathrm{d}t} + \frac{(\bar{z}_1-\bar{z}_0)}{\Gamma_1}\frac{\mathrm{d}\Gamma_1}{\mathrm{d}t} = \lim_{z\to z_1}[\frac{\mathrm{d}}{\mathrm{d}z}(F + \frac{\mathrm{i}\Gamma_1}{2\pi}\log(z-z_1))]. \quad (52)$$

with initial conditions $\Gamma_1(0) = 0$, $\zeta_1(0) = 1$. The governing equation in the mapped plane (see appendix B.2) is

$$\begin{aligned} &\frac{\dot{\bar{\zeta_1}}}{2}(1-\frac{1}{\bar{\zeta_1^2}}) + \frac{(1-\bar{\zeta_1})^2}{2\bar{\zeta_1}(\bar{\zeta_1^2}-\zeta_1^2)}(2\zeta_1\dot{\zeta_1}\frac{1-\bar{\zeta_1^2}}{1-\zeta_1^2} - 2\bar{\zeta_1}\dot{\bar{\zeta_1}}\frac{1-\zeta_1^2}{1-\bar{\zeta_1^2}}) \\ &= \frac{2\zeta_1^2}{\zeta_1^2-1}[\frac{Q}{2\pi\zeta_1} + \frac{\mathrm{i}\Gamma_1(t)}{2\pi}(\frac{1}{\zeta_1-\bar{\zeta_1}} + \frac{1}{\zeta_1+\bar{\zeta_1}} - \frac{1}{2\zeta_1}) + \frac{\mathrm{i}\Gamma_1(t)}{2\pi\zeta_1(\zeta_1^2-1)}]. \quad (53)\end{aligned}$$

We solve the equation numerically in the next section.

4.2 Numerical solution

Let $\zeta = \xi + \mathrm{i}\eta$ in (53). We obtain two equations (see appendix F) from the real part and the imaginary part respectively and then solve them numerically by using “NDSolve” in Mathematica. We also use Mathematica to plot the solution,

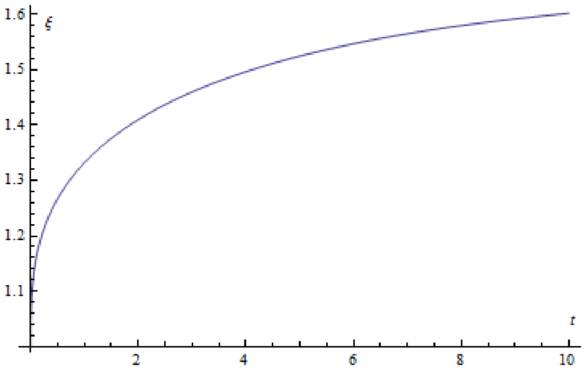

Figure 18: numerical solution of $\xi(t)$.

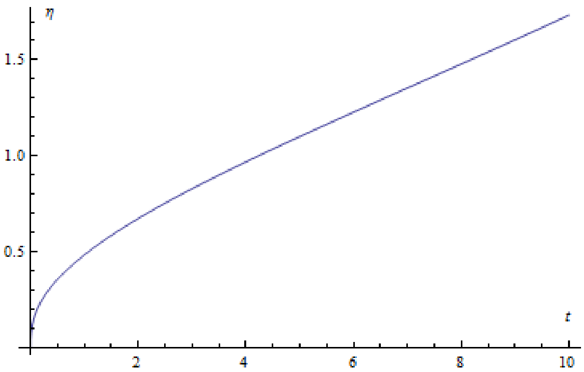

Figure 19: numerical solution of $\eta(t)$.

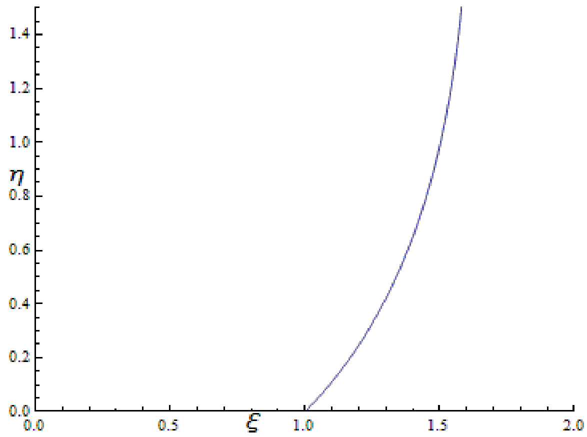

Figure 20: trajectory of ζ_1 in the mapped plane.

By symmetry, we can easily solve ζ_1^* and get,

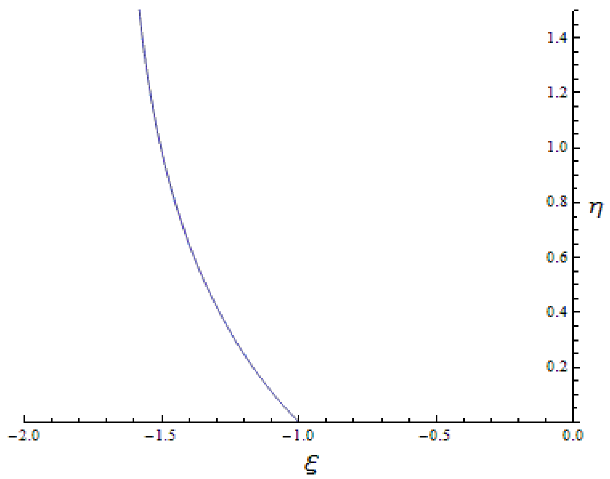

Figure 21: trajectory of ζ_1^* in the mapped plane.

Now we plot the solution of ζ_1 and the solution of ζ_1^* in the same graph as below,

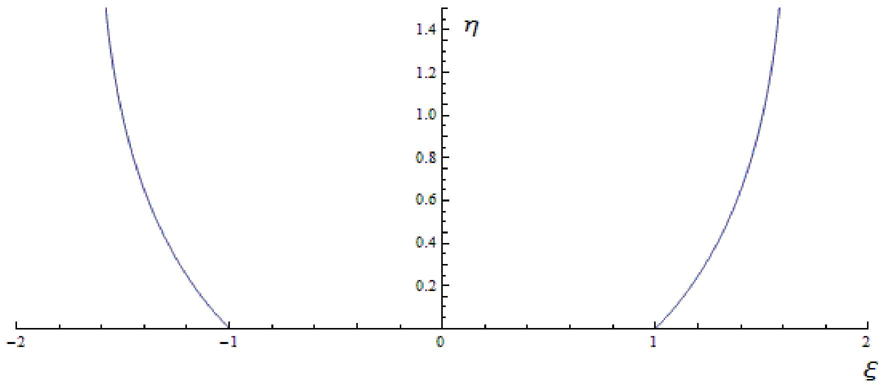

Figure 22: numerical solution of ζ_1 and ζ_1^*.

We transform the solution back into the physical z-plane by using the conformal mapping (49). Again we use Mathematica to plot the graphs,

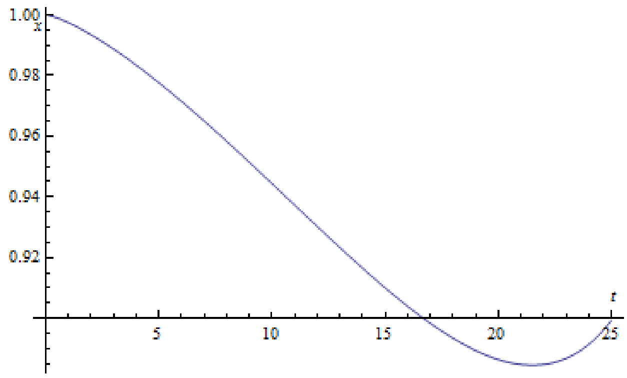

Figure 23: numerical solution of $x(t)$.

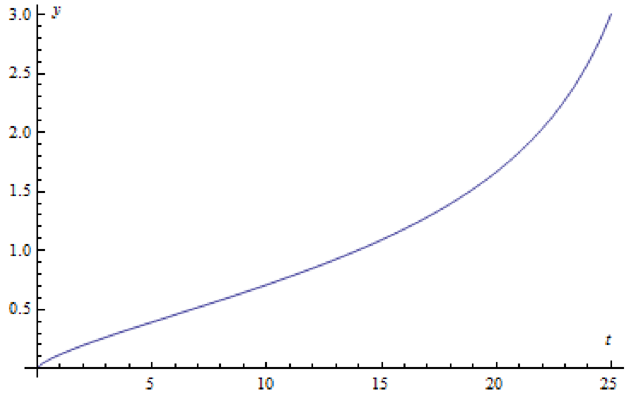

Figure 24: numerical solution of $y(t)$.

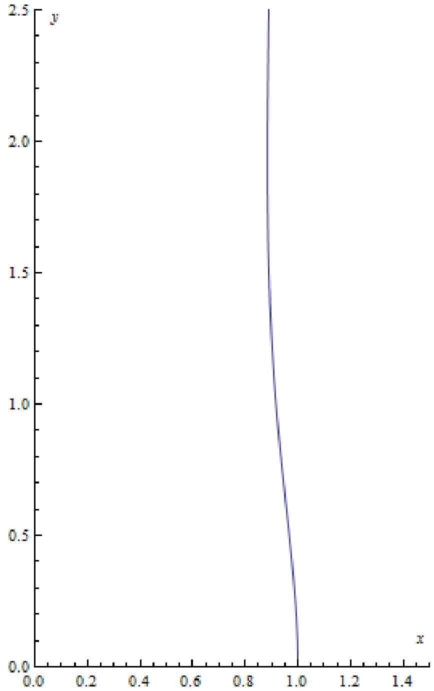

Figure 25: Trajectory of the vortex in the physical plane.

By symmetry, we can easily get the solution of the vortex on the other side. Then we put the two vortices in the same graph as the following,

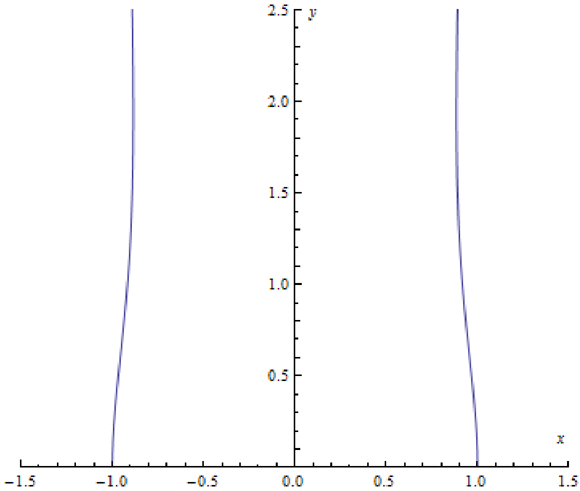

Figure 26: Trajectories of the two vortices in the physical plane.

5 Conclusion

We have seen three typical examples of flows past sharp corner. For the semi-infinite plate problem, we used Brown-Michael equation as the governing equation to figure out the position of the starting vortex. And we solved it both analytically and numerically. It is pleasant to see the coincidence of the numerical solution and the exact solution. For the finite wedge problem, we used the similar technique and method as we did in the semi-infinite case to investigate the starting vortex. We obtain that the vortex always follows a straight line angled $\frac{\pi}{4}$ with the axis in the mapped plane. The result is quite significant. Finally we studied the problem that flows pass two semi-infinite plates with a single gap. This time there are two starting vortices generated which are symmetric about $\mathrm{Re}(z) = 0$. We rewrite Brown-Michael equation in this case and got it solved numerically. We draw the figure of the trajectories as well to show what happen to the vortices in the physical z-plane.

A Brown-Michael Model

In this section we derive the Brown-Michael equation based on complex method by using a conservation of momentum. Consider a general contour C moving with a velocity $\mathbf{u}_c$ and enclosing fluid only, then

$$\dot{\mathbf{M}} = -\int_C [p + \rho\mathbf{u}(\mathbf{u} - \mathbf{u}_c)]\mathbf{n}\mathrm{d}l \tag{54}$$

where $\mathbf{M}$ is the momentum in C and the right hand side is the total force applied which is the sum of force given by the outside fluid on C and the flux of momentum through C. Now we use the complex notation. We denote F as complex potential. The complex velocity is $w = \frac{\mathrm{d}F}{\mathrm{d}z} = u - \mathrm{i}v$. (So $\mathbf{u} \equiv \bar{w}$) Note that $\mathbf{n}.\mathrm{d}l = (-\mathrm{d}y, \mathrm{d}x)$. In the complex plane, we can write it as $\mathbf{n}.\mathrm{d}l \equiv \mathrm{i}\mathrm{d}z$. Then we have the following equality

$$\mathbf{u}.\mathbf{n}\mathrm{d}l = -u\mathrm{d}y + v\mathrm{d}x = \mathrm{Re}[(u - \mathrm{i}v)(\mathrm{i}\mathrm{d}x - \mathrm{d}y)] = \mathrm{Re}[\mathrm{i}w\mathrm{d}z]$$

by Bernoulli Equation, we have

$$p(t) = p_0(t) - \frac{\rho}{2}(F_t + \bar{F}_t + w\bar{w}) \tag{55}$$

Substitute this back into (1) and get

$$\dot{\mathbf{M}} = -\int_C [p_0(t) - \rho(\frac{F_t}{2} + \frac{\bar{F}_t}{2} + \frac{w\bar{w}}{2} - \mathbf{u}(\mathbf{u} - \mathbf{u}_c))]\mathrm{i}\mathrm{d}z$$

As p_0 is analytic everywhere within the contour C, the line integral $\int_C p_0(t)\mathrm{d}z$ equals zero by Cauchy theorem. So

$$\begin{aligned}
\dot{\mathbf{M}} &= \int_C \mathrm{i}\rho[(\frac{F_t}{2} + \frac{\bar{F}_t}{2} + \frac{w\bar{w}}{2} - \mathbf{u}(\mathbf{u} - \mathbf{u}_c)]\mathrm{d}z \\
&= \int_C \frac{\mathrm{i}\rho}{2}(F_t + \bar{F}_t)\mathrm{d}z + \int_C \frac{\mathrm{i}\rho}{2} w\bar{w}\ \mathrm{d}z - \int_C \rho\bar{w}.\mathrm{Re}[(w - w_c)\mathrm{i}\mathrm{d}z] \\
&= \int_C \frac{\mathrm{i}\rho}{2}(F_t + \bar{F}_t)\mathrm{d}z + \int_C \frac{\mathrm{i}\rho}{2} w\bar{w}\ \mathrm{d}z - \int_C \frac{\mathrm{i}\rho\bar{w}}{2}[(w - w_c) - (\bar{w} - \bar{w}_c)]\mathrm{d}z \\
&= \int_C \frac{\mathrm{i}\rho}{2}(F_t + \bar{F}_t)\mathrm{d}z + \underline{\int_C \frac{\mathrm{i}\rho}{2}\bar{w}w_c\mathrm{d}z} + \underline{\int_C \frac{\mathrm{i}\rho\bar{w}}{2}(\bar{w} - \bar{w}_c)\mathrm{d}z} \\
&= \int_C \frac{\mathrm{i}\rho}{2}(F_t + \bar{F}_t)\mathrm{d}z - \overline{\frac{\mathrm{i}\rho}{2}\bar{w}_c\int_C w\mathrm{d}\bar{z}} - \overline{\frac{\mathrm{i}\rho}{2}\int_C w(w - w_c)\mathrm{d}z}
\end{aligned}$$

where we replaced $\mathbf{u}(\mathbf{u}-\mathbf{u}_c)\mathbf{n}\mathrm{d}l$ by $\bar{w}.\mathrm{Re}[(w-w_c)\mathrm{i}\mathrm{d}z]$. Thus the equation of change of the momentum is written as

$$\dot{\mathbf{M}} = \int_C \frac{\mathrm{i}\rho}{2}(F_t + \bar{F}_t)\mathrm{d}z - \overline{\frac{\mathrm{i}\rho}{2}\bar{w}_c \int_C w\mathrm{d}\bar{z}} - \overline{\frac{\mathrm{i}\rho}{2}\int_C w(w-w_c)\mathrm{d}z}. \qquad (56)$$

Now we decompose the complex potential and the complex velocity as

$$\begin{aligned} F &= \frac{\Gamma_n}{2\mathrm{i}\pi}\log(z-z_n) + \tilde{F}_n(z), \\ \omega &= \frac{\Gamma_n}{2\mathrm{i}\pi}\frac{1}{(z-z_n)} + \tilde{\omega}_n(z). \end{aligned}$$

$\tilde{F}_n$ and $\tilde{\omega}_n$ are respectively the desingularised complex potential and velocity of the fluid at the vortex position. The last term in (56) can be easily calculated as below

$$\begin{aligned} \int_C w(w-w_c)\mathrm{d}z &= \int_C w^2\mathrm{d}z - w_c\int_C w\mathrm{d}z \\ &= \int_C (\frac{\Gamma_n}{2\mathrm{i}\pi}\frac{1}{(z-z_n)} + \tilde{\omega}_n(z))^2\mathrm{d}z - w_c\Gamma_n \\ &= \int_C [(\frac{\Gamma_n}{2\mathrm{i}\pi})^2\frac{1}{(z-z_n)^2} + \tilde{\omega}_n(z)^2 + \frac{\Gamma_n}{2\mathrm{i}\pi}\frac{2\tilde{\omega}_n}{z-z_n}]\mathrm{d}z - w_c\Gamma_n \\ &= 2\Gamma_n\tilde{\omega}_n - w_c\Gamma_n. \end{aligned}$$

where we used the theorem of residues to compute the contour integral. To get the second term, we need to integrate $\int_C w\mathrm{d}\bar{z}$. Obviously its value is zero since w is analytic everywhere except at the vortex point which is not included in the conjugate position of contour C. Therefore we have $\overline{\frac{\mathrm{i}\rho}{2}\bar{w}_c\int_C w\mathrm{d}\bar{z}} = 0$. The rest of the work is to get rid of $\int_C(F_t + \bar{F}_t)\mathrm{d}z$. We know that

$$\begin{aligned} F_t + \bar{F}_t &= \frac{\dot{\Gamma}_n}{2\mathrm{i}\pi}\log(z-z_n) - \frac{\Gamma_n}{2\mathrm{i}\pi}\frac{\dot{z}_n}{z-z_n} + \overline{\frac{\dot{\Gamma}_n}{2\mathrm{i}\pi}\log(z-z_n)} - \overline{\frac{\Gamma_n}{2\mathrm{i}\pi}\frac{\dot{z}_n}{z-z_n}} \\ &= \frac{\dot{\Gamma}_n}{\pi}\mathrm{Re}[-\mathrm{i}\log(z-z_n)] - \frac{\Gamma_n}{2\mathrm{i}\pi}\frac{\dot{z}_n}{z-z_n} - \overline{\frac{\Gamma_n}{2\mathrm{i}\pi}\frac{\dot{z}_n}{z-z_n}}. \end{aligned}$$

Following from that, we decompose C as three parts: two straight lines and one circle.

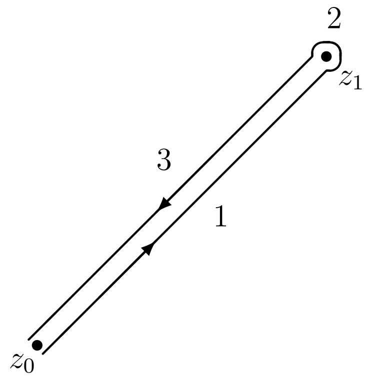

Figure 25: decomposition of the contour.

$z = z_{n,0} + re^{\mathrm{i}\alpha}$ on 1 and $z = z_{n,0} + re^{\mathrm{i}(\alpha+2\pi)}$ on 3, where $0 \le r \le |z_n - z_{n,0}|$ and α is constant. On 2 , $z = z_n + \epsilon e^{\mathrm{i}\theta}$ where $0 \le \theta \le 2\pi$. Using these equalities, we can easily obtain that

$$\begin{aligned}\int_1 \mathrm{Re}[-\mathrm{i}\log(z-z_n)]\mathrm{d}z &= \int_1 \mathrm{Re}[-\mathrm{i}(\log|z-z_n| + \mathrm{i}\arg(z-z_n))]\mathrm{d}z\\ &= \int_1 \arg(z-z_n)\mathrm{d}z\\ &= \int_{z_{n,0}}^{z_n} \alpha\mathrm{d}z = \alpha(z_n - z_{n,0}).\end{aligned}$$

And similarly,

$$\begin{aligned}\int_3 \mathrm{Re}[-\mathrm{i}\log(z-z_n)]\mathrm{d}z &= \int_3 \arg(z-z_n)\mathrm{d}z\\ &= \int_{z_n}^{z_{n,0}} (2\pi+\alpha)\mathrm{d}z\\ &= -(2\pi+\alpha)(z_n - z_{n,0}).\end{aligned}$$

The integral over route 2 tends to zero as ϵ goes to zero since

$$\begin{aligned}\int_2 \mathrm{Re}[-\mathrm{i}\log(z-z_n)]\mathrm{d}z &= \int_2 \mathrm{Re}[\epsilon\theta]\mathrm{d}z\\ &= \epsilon\int_2 \theta\mathrm{d}z \to 0 \text{ as } \epsilon \to 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_C (F_t + \bar{F}_t)\mathrm{d}z &= \int_C \frac{\dot{\Gamma}_n}{\pi}\mathrm{Re}[-\mathrm{i}\log(z - z_n)]\mathrm{d}z - \int_C \frac{\Gamma_n}{2\mathrm{i}\pi}\frac{\dot{z}_n}{z - z_n}\mathrm{d}z - \int_C \overline{\frac{\Gamma_n}{2\mathrm{i}\pi}\frac{\dot{z}_n}{z - z_n}}\mathrm{d}z \\ &= \frac{\dot{\Gamma}_n}{\pi}(\alpha - 2\pi - \alpha)(z_n - z_{n,0}) - \Gamma_n\dot{z}_n - 0 \\ &= -2\dot{\Gamma}_n(z_n - z_{n,0}) - \Gamma_n\dot{z}_n.\end{aligned}$$

Summary: We have the following equations so far,

$$\int_C w(w - w_c)\mathrm{d}z = 2\Gamma_n\tilde{\omega}_n - w_c\Gamma_n, \tag{57}$$

$$\int_C (F_t + \bar{F}_t)\mathrm{d}z = -2\dot{\Gamma}_n(z_n - z_{n,0}) - \Gamma_n\dot{z}_n. \tag{58}$$

Using the results (57) and (58) in (56), we get

$$\begin{aligned}\dot{\mathbf{M}} &= \frac{\mathrm{i}\rho}{2}(-2\dot{\Gamma}_n(z_n - z_{n,0}) - \Gamma_n\dot{z}_n) - 0 - \overline{\frac{\mathrm{i}\rho}{2}(2\Gamma_n\tilde{\omega}_n - w_c\Gamma_n)} \\ &= -\mathrm{i}\rho(\dot{\Gamma}_n(z_n - z_{n,0}) + \frac{\Gamma_n\dot{z}_n}{2} - \Gamma_n\overline{\tilde{\omega}_n} + \frac{\Gamma_n\overline{w}_c}{2}) \\ &= -\mathrm{i}\rho(\dot{\Gamma}_n(z_n - z_{n,0}) + \frac{\Gamma_n\dot{z}_n}{2} - \Gamma_n\overline{\tilde{\omega}_n} + \frac{\Gamma_n\dot{z}_n}{2}) \\ &= -\mathrm{i}\rho(\dot{\Gamma}_n(z_n - z_{n,0}) + \Gamma_n\dot{z}_n - \Gamma_n\overline{\tilde{\omega}_n}).\end{aligned}$$

where we used the velocity of the contour $w_c = \dot{z}_n$ near the vortex. As $\epsilon \to 0$, the contour shrinks down and the change of the momentum becomes zero. Hence $\dot{\mathbf{M}} = 0$. Then we finally obtain the so-called Brown-Michael equation.

$$\dot{\Gamma}_n(z_n - z_{n,0}) + \Gamma_n\dot{z}_n = \Gamma_n\overline{\tilde{\omega}_n}$$

Or

$$\frac{\dot{\Gamma}_n}{\Gamma_n}(z_n - z_{n,0}) + \dot{z}_n = \overline{\tilde{\omega}_n} \tag{59}$$

B Brown-Michael equation in the mapped planes

B.1 Flows past a semi-infinite plate

Recall that the conformal mapping is $z = -\mathrm{i}\zeta^2$. We rewrite the right hand side of (4),

$$\begin{aligned}
F - \frac{\mathrm{i}\Gamma_1}{2\pi}\log(z - z_1) &= F - \frac{\mathrm{i}\Gamma_1}{2\pi}\log[z(\zeta) - z(\zeta_1)] \\
&= F - \frac{\mathrm{i}\Gamma_1}{2\pi}\log[z(\zeta) - (z(\zeta) + (\zeta_1 - \zeta)\frac{\mathrm{d}z}{\mathrm{d}\zeta}|_{\zeta=\zeta_1} \\
&\qquad + \frac{(\zeta_1 - \zeta)^2}{2}\frac{\mathrm{d}^2 z}{\mathrm{d}\zeta^2}|_{\zeta=\zeta_1}) + \text{h.o.t.}] \\
&= F - \frac{\mathrm{i}\Gamma_1}{2\pi}\{\log[\zeta - \zeta_1] + \log[\frac{\mathrm{d}z}{\mathrm{d}\zeta}|_{\zeta=\zeta_1}] \\
&\qquad + \log[1 + \frac{(\zeta - \zeta_1)}{2}(\frac{\mathrm{d}^2 z}{\mathrm{d}\zeta^2}\frac{\mathrm{d}z}{\mathrm{d}\zeta}^{-1})_{\zeta=\zeta_1} + \text{h.o.t.})]\} \\
&= F - \frac{\mathrm{i}\Gamma_1}{2\pi}\log[\zeta - \zeta_1] - \frac{\mathrm{i}\Gamma_1}{2\pi}\log[\frac{\mathrm{d}z}{\mathrm{d}\zeta}|_{\zeta=\zeta_1}] - \frac{\mathrm{i}\Gamma_1}{4\pi}(\zeta - \zeta_1)(\frac{\mathrm{d}^2 z}{\mathrm{d}\zeta^2}\frac{\mathrm{d}z}{\mathrm{d}\zeta}^{-1})_{\zeta=\zeta_1} \\
&\qquad + \text{h.o.t.}
\end{aligned}$$

Differentiate it with respect to z and take the limit as $z \to z_1$, the higher order term tends to zero since it is of order $O(\zeta - \zeta_1)$. Note that $\frac{\mathrm{i}\Gamma_1}{2\pi}\log[\frac{\mathrm{d}z}{\mathrm{d}\zeta}|_{\zeta=\zeta_1}]$

will vanish after the differentiation. Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}[F-\frac{\mathrm{i}\Gamma_1}{2\pi}\log(z-z_1)] &= \frac{\mathrm{d}F}{\mathrm{d}z}-\frac{\mathrm{i}\Gamma_1}{2\pi(\zeta-\zeta_1)}\frac{\mathrm{d}\zeta}{\mathrm{d}z}-\frac{\mathrm{i}\Gamma_1}{4\pi}\frac{\mathrm{d}\zeta}{\mathrm{d}z}[(\frac{\mathrm{d}^2z}{\mathrm{d}\zeta^2})(\frac{\mathrm{d}z}{\mathrm{d}\zeta})^{-1}]_{\zeta=\zeta_1}+\text{h.o.t.}\\ &= \frac{\mathrm{d}\zeta}{\mathrm{d}z}[\frac{\mathrm{d}F}{\mathrm{d}\zeta}-\frac{\mathrm{i}\Gamma_1}{2\pi(\zeta-\zeta_1)}-\frac{\mathrm{i}\Gamma_1}{4\pi}\frac{1}{\zeta_1}]+\text{h.o.t.}\\ &= \frac{\mathrm{d}\zeta}{\mathrm{d}z}[U(t)+\frac{\mathrm{i}\Gamma_1}{2\pi}(\frac{1}{\zeta-\zeta_1}-\frac{1}{\zeta-\bar{\zeta}_1})-\frac{\mathrm{i}\Gamma_1}{2\pi(\zeta-\zeta_1)}+\\ &\qquad \sum_{n=2}^{N}\frac{\mathrm{i}\Gamma_n}{2\pi}(\frac{1}{\zeta-\zeta_n}-\frac{1}{\zeta-\bar{\zeta}_n})-\frac{\mathrm{i}\Gamma_1}{4\pi}\frac{1}{\zeta_1}]+\text{h.o.t.}\\ &= \frac{-1}{2\mathrm{i}\zeta_1}[U(t)-\frac{\mathrm{i}\Gamma_1}{2\pi}\frac{1}{\zeta-\bar{\zeta}_1}-\frac{\mathrm{i}\Gamma_1}{4\pi}\frac{1}{\zeta_1}+\\ &\qquad \sum_{n=2}^{N}\frac{\mathrm{i}\Gamma_n}{2\pi}(\frac{1}{\zeta-\zeta_n}-\frac{1}{\zeta-\bar{\zeta}_n})]+\text{h.o.t.}\end{aligned}$$

Take the limit $\zeta \rightarrow \zeta_1$, we get

$$\begin{aligned}\lim_{\zeta\rightarrow\zeta_1}\frac{\mathrm{d}}{\mathrm{d}z}[F-\frac{\mathrm{i}\Gamma_1}{2\pi}\log(z-z_1)] = \frac{-1}{2\mathrm{i}\zeta_1}[U(t)-\frac{\mathrm{i}\Gamma_1}{2\pi}\frac{1}{\zeta_1-\bar{\zeta}_1}-\frac{\mathrm{i}\Gamma_1}{4\pi}\frac{1}{\zeta_1}\\ +\sum_{n=2}^{N}\frac{\mathrm{i}\Gamma_n}{2\pi}(\frac{1}{\zeta_1-\zeta_n}-\frac{1}{\zeta_1-\bar{\zeta}_n})].\end{aligned} \quad (60)$$

The above result is valid for any $r = 2, 3, ..., N$. Note that the term $-\frac{\mathrm{i}\Gamma_1}{4\pi}[(\frac{\mathrm{d}^2z}{\mathrm{d}\zeta^2})(\frac{\mathrm{d}z}{\mathrm{d}\zeta})^{-1}]_{\zeta=\zeta_1}$ is called the Routh correction. Using the mapping function $z = -\mathrm{i}\zeta^2$, the L.H.S. of Brown-Michael equation (4) is

$$\text{L.H.S.} = 2\mathrm{i}\bar{\zeta}_1\frac{\mathrm{d}\bar{\zeta}_1}{\mathrm{d}t}+\frac{\mathrm{i}\bar{\zeta}_1^2}{\Gamma_1}\frac{\mathrm{d}\Gamma_1}{\mathrm{d}t}. \quad (61)$$

And the R.H.S. is

$$\text{R.H.S.} = \frac{-1}{2\mathrm{i}\zeta_1}[U(t)-\frac{\mathrm{i}\Gamma_1}{2\pi}\frac{1}{\zeta_1-\bar{\zeta}_1}-\frac{\mathrm{i}\Gamma_1}{4\pi}\frac{1}{\zeta_1}+\sum_{n=2}^{N}\frac{\mathrm{i}\Gamma_n}{2\pi}(\frac{1}{\zeta_1-\zeta_n}-\frac{1}{\zeta_1-\bar{\zeta}_n})]. \quad (62)$$

Need to calculate $\frac{1}{\Gamma_1}\frac{\mathrm{d}\Gamma_1}{\mathrm{d}t}$,

$$\begin{aligned}\frac{1}{\Gamma_1}\frac{\mathrm{d}\Gamma_1}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(\log\Gamma_1) = \frac{\mathrm{d}}{\mathrm{d}t}[\log(2\pi\mathrm{i}(\frac{\zeta_1\bar{\zeta}_1}{\zeta_1-\bar{\zeta}_1})(U+\sum_{n=2}^{N}\frac{\mathrm{i}\Gamma_n}{2\pi}(\frac{\zeta_n\bar{\zeta}_n}{\zeta_n-\bar{\zeta}_n})))]\\
&= \frac{\dot{\zeta}_1}{\zeta_1}+\frac{\dot{\bar{\zeta}}_1}{\bar{\zeta}_1}-\frac{\dot{\zeta}_1-\dot{\bar{\zeta}}_1}{\zeta_1-\bar{\zeta}_1}+\\
&\qquad \frac{1}{U+\sum_{n=2}^{N}\frac{\mathrm{i}\Gamma_n}{2\pi}(\frac{\zeta_n\bar{\zeta}_n}{\zeta_n-\bar{\zeta}_n})}[\dot{U}+\sum_{n=2}^{N}\frac{\mathrm{i}\Gamma_n}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{\bar{\zeta}_n}-\frac{1}{\zeta_n})]\\
&= \frac{-\dot{\zeta}_1\bar{\zeta}_1}{\zeta_1(\zeta_1-\bar{\zeta}_1)}+\frac{\dot{\bar{\zeta}}_1\zeta_1}{\bar{\zeta}_1(\zeta_1-\bar{\zeta}_1)}+\\
&\qquad \frac{1}{U+\sum_{n=2}^{N}\frac{\mathrm{i}\Gamma_n}{2\pi}(\frac{\zeta_n\bar{\zeta}_n}{\zeta_n-\bar{\zeta}_n})}[\dot{U}+\sum_{n=2}^{N}\frac{\mathrm{i}\Gamma_n}{2\pi}(\frac{\dot{\zeta}_n}{\zeta_n^2}-\frac{\dot{\bar{\zeta}}_n}{\bar{\zeta}_n^2})].\end{aligned}$$

The L.H.S. becomes

$$\begin{aligned}\text{L.H.S.} &= 2\mathrm{i}\bar{\zeta}_1\dot{\bar{\zeta}}_1+\frac{-\mathrm{i}\dot{\zeta}_1\bar{\zeta}_1^3}{\zeta_1(\zeta_1-\bar{\zeta}_1)}+\frac{\mathrm{i}\dot{\bar{\zeta}}_1\zeta_1\bar{\zeta}_1}{(\zeta_1-\bar{\zeta}_1)}+\\
&\qquad \frac{\mathrm{i}\bar{\zeta}_1^2}{U+\sum_{n=2}^{N}\frac{\mathrm{i}\Gamma_n}{2\pi}(\frac{\zeta_n\bar{\zeta}_n}{\zeta_n-\bar{\zeta}_n})}[\dot{U}+\sum_{n=2}^{N}\frac{\mathrm{i}\Gamma_n}{2\pi}(\frac{\dot{\zeta}_n}{\zeta_n^2}-\frac{\dot{\bar{\zeta}}_n}{\bar{\zeta}_n^2})]. \quad (63)\end{aligned}$$

Finally we put (62) and (63) together to obtain equation (6),

$$\begin{aligned}&(2\mathrm{i}\bar{\zeta}_1+\frac{\mathrm{i}\zeta_1\bar{\zeta}_1}{\zeta_1-\bar{\zeta}_1})\frac{\mathrm{d}\bar{\zeta}_1}{\mathrm{d}t}-(\frac{\mathrm{i}\bar{\zeta}_1^3}{\zeta_1(\zeta_1-\bar{\zeta}_1)})\frac{\mathrm{d}\zeta_1}{\mathrm{d}t}\\
&\quad = \frac{\mathrm{i}}{2\zeta_1}(U-\frac{\mathrm{i}\Gamma_1}{2\pi(\zeta_1-\bar{\zeta}_1)}+\sum_{n=2}^{N}\frac{\mathrm{i}\Gamma_n}{2\pi}\frac{(\zeta_n-\bar{\zeta}_n)}{(\zeta_1-\zeta_n)(\zeta_1-\bar{\zeta}_n)}-\frac{\mathrm{i}\Gamma_1}{4\pi\zeta_1})\\
&\qquad -\mathrm{i}\bar{\zeta}_1^2[\frac{\mathrm{d}U}{\mathrm{d}t}+\sum_{n=2}^{N}\frac{\mathrm{i}\Gamma_n}{2\pi}(\frac{1}{\zeta_n^2}\frac{\mathrm{d}\zeta_n}{\mathrm{d}t}-\frac{\mathrm{d}\bar{\zeta}_n}{\mathrm{d}t})](U+\sum_{n=2}^{N}\frac{\mathrm{i}\Gamma_n}{2\pi}\frac{\zeta_n-\bar{\zeta}_n}{\zeta_n\bar{\zeta}_n})^{-1}.\end{aligned}$$

Similarly, we can get the equations for any $n = 2, 3, \ldots N$ as below

$$\frac{\mathrm{d}\bar{\zeta}_r}{\mathrm{d}t} = \frac{1}{4\zeta_r\bar{\zeta}_r}(U-\frac{\mathrm{i}\Gamma_r}{2\pi(\zeta_r-\bar{\zeta}_r)}+\sum_{n\neq r}\frac{\mathrm{i}\Gamma_n}{2\pi}\frac{(\zeta_n-\bar{\zeta}_n)}{(\zeta_1-\zeta_n)(\zeta_1-\bar{\zeta}_n)}-\frac{\mathrm{i}\Gamma_r}{4\pi\zeta_r}).$$

which is equation (7) defined previously.

B.2 Flows past two semi-infinite plates with a single gap

Recall that the conformal mapping is defined as $z = \frac{1}{2}(\zeta + \zeta^{-1})$. So

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \frac{1}{2}(1 - \zeta^{-2}). \tag{64}$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\zeta^2} = \zeta^{-3}. \tag{65}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}z}(F + \frac{\mathrm{i}\Gamma_1}{2\pi}\log(z - z_1))] = (\frac{\mathrm{d}z}{\mathrm{d}\zeta})^{-1}(\frac{\mathrm{d}F}{\mathrm{d}\zeta} + \frac{\mathrm{i}\Gamma_1}{2\pi(\zeta - \zeta_1)} + \frac{\mathrm{i}\Gamma_1}{4\pi}\frac{\frac{\mathrm{d}^2 z}{\mathrm{d}\zeta^2}(\zeta_1)}{\frac{\mathrm{d}z}{\mathrm{d}\zeta}(\zeta_1)})$$

Therefore

$$\begin{aligned}
&\lim_{\zeta \to \zeta_1} \frac{\mathrm{d}}{\mathrm{d}z}[F + \frac{\mathrm{i}\Gamma_1}{2\pi}\log(z - z_1)] = \\
&= \frac{2\zeta_1^2}{\zeta_1^2 - 1}[\frac{Q}{2\pi\zeta_1} + \frac{\mathrm{i}\Gamma_1}{2\pi}(\frac{1}{\zeta_1 - \bar{\zeta}_1} + \frac{1}{\zeta_1 + \bar{\zeta}_1} - \frac{1}{2\zeta_1}) + \frac{\mathrm{i}\Gamma_1}{2\pi\zeta_1(\zeta_1^2 - 1)}]
\end{aligned} \tag{66}$$

And

$$\frac{\mathrm{d}\bar{z}_1}{\mathrm{d}t} + \frac{(\bar{z}_1 - \bar{z}_0)}{\Gamma_1}\frac{\mathrm{d}\Gamma_1}{\mathrm{d}t} = \frac{\dot{\bar{\zeta}}_1}{2}(1 - \frac{1}{\bar{\zeta}_1^2}) + \frac{1}{2}(\bar{\zeta}_1 + \frac{1}{\bar{\zeta}_1} - 2)\frac{\dot{\Gamma}_1}{\Gamma_1} \tag{67}$$

We need to calculate $\frac{\dot{\Gamma}_1}{\Gamma_1}$,

$$\begin{aligned}
\frac{\dot{\Gamma}_1}{\Gamma_1} &= \frac{\mathrm{d}}{\mathrm{d}t}(\log(\Gamma_1)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(\log(1 - \zeta_1^2) + \log(1 - \bar{\zeta}_1^2) - \log(\bar{\zeta}_1^2 - \zeta_1^2)) \\
&= -\frac{2\zeta_1\dot{\zeta}_1}{1 - \zeta_1^2} - \frac{2\bar{\zeta}_1\dot{\bar{\zeta}}_1}{1 - \bar{\zeta}_1^2} + \frac{2\zeta_1\dot{\zeta}_1 - 2\bar{\zeta}_1\dot{\bar{\zeta}}_1}{\bar{\zeta}_1^2 - \zeta_1^2} \\
&= 2\zeta_1\dot{\zeta}_1(\frac{1}{\bar{\zeta}_1^2 - \zeta_1^2} - \frac{1}{1 - \zeta_1^2}) - 2\bar{\zeta}_1\dot{\bar{\zeta}}_1(\frac{1}{\bar{\zeta}_1^2 - \zeta_1^2} + \frac{1}{1 - \bar{\zeta}_1^2}) \\
&= 2\zeta_1\dot{\zeta}_1\frac{1 - \bar{\zeta}_1^2}{(\bar{\zeta}_1^2 - \zeta_1^2)(1 - \zeta_1^2)} - 2\bar{\zeta}_1\dot{\bar{\zeta}}_1\frac{1 - \zeta_1^2}{(\bar{\zeta}_1^2 - \zeta_1^2)(1 - \bar{\zeta}_1^2)}
\end{aligned}$$

The Brown-Michael equation in the mapped plane is as the following

$$\frac{\dot{\bar{\zeta}}_1}{2}(1-\frac{1}{\bar{\zeta}_1^2})+\frac{(1-\bar{\zeta}_1)^2}{2\bar{\zeta}_1(\bar{\zeta}_1^2-\zeta_1^2)}(2\zeta_1\dot{\zeta}_1\frac{1-\bar{\zeta}_1^2}{1-\zeta_1^2}-2\bar{\zeta}_1\dot{\bar{\zeta}}_1\frac{1-\zeta_1^2}{1-\bar{\zeta}_1^2})$$
$$=\frac{2\zeta_1^2}{\zeta_1^2-1}[\frac{Q}{2\pi\zeta_1}+\frac{\mathrm{i}\Gamma_1(t)}{2\pi}(\frac{1}{\zeta_1-\bar{\zeta}_1}+\frac{1}{\zeta_1+\bar{\zeta}_1}-\frac{1}{2\zeta_1})+\frac{\mathrm{i}\Gamma_1(t)}{2\pi\zeta_1(\zeta_1^2-1)}] \quad (68)$$

C Transformation to polar form

Let $\zeta_1 = \rho_1 e^{\mathrm{i}(\frac{\pi}{2}-\theta_1)} = \mathrm{i}\rho_1 e^{-\mathrm{i}\theta_1}$. $\bar{\zeta}_1 = -\mathrm{i}\rho_1 e^{\mathrm{i}\theta}$. The derivatives with with respect to t are $\dot{\zeta}_1 = \zeta_1(\frac{\dot{\rho}_1}{\rho_1}-\mathrm{i}\theta_1)$, $\dot{\bar{\zeta}}_1 = \bar{\zeta}_1(\frac{\dot{\rho}_1}{\rho_1}+\mathrm{i}\theta_1)$. Note that

$$2\mathrm{i}\bar{\zeta}_1+\frac{\mathrm{i}\zeta_1\bar{\zeta}_1}{\zeta_1-\bar{\zeta}_1}=3\mathrm{i}\bar{\zeta}_1+\frac{\mathrm{i}\bar{\zeta}_1^3}{\bar{\zeta}_1(\zeta_1-\bar{\zeta}_1)} \quad (\dagger)$$

Using (†),

$$\begin{aligned}
\text{L.H.S. of (8)} &= (3\mathrm{i}\bar{\zeta}_1+\frac{\mathrm{i}\bar{\zeta}_1^3}{\bar{\zeta}_1(\zeta_1-\bar{\zeta}_1)})\dot{\bar{\zeta}}_1-\frac{\mathrm{i}\zeta_1^3}{\bar{\zeta}_1(\zeta_1-\bar{\zeta}_1)}\dot{\zeta}_1 \\
&= 3\mathrm{i}\bar{\zeta}_1\dot{\bar{\zeta}}_1+\frac{\mathrm{i}\zeta_1^3}{(\zeta_1-\bar{\zeta}_1)}(\frac{\dot{\bar{\zeta}}_1}{\bar{\zeta}_1}-\frac{\dot{\zeta}_1}{\zeta_1}) \\
&= \mathrm{i}\bar{\zeta}_1^2(\frac{3\dot{\rho}_1}{\rho_1}+3\mathrm{i}\theta_1+\frac{\bar{\zeta}_1\dot{\theta}_1}{\rho_1\cos\theta_1}) \\
&= \mathrm{i}\bar{\zeta}_1^2(\frac{3\dot{\rho}_1}{\rho_1}+3\mathrm{i}\theta_1-\frac{\mathrm{i}\dot{\theta}_1}{\cos\theta_1}(\cos\theta_1+\mathrm{i}\sin\theta_1)) \\
&= \mathrm{i}\bar{\zeta}_1^2(\frac{3\dot{\rho}_1}{\rho_1}+2\mathrm{i}\theta_1+\dot{\theta}_1\tan\theta_1).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
\text{R.H.S. of (8)} &= \frac{\mathrm{i}}{2\zeta_1}(U - \frac{\mathrm{i}\Gamma_1}{2\pi(\zeta_1 - \bar{\zeta}_1)} - \frac{\mathrm{i}\Gamma_1}{4\pi\zeta_1}) - \frac{\mathrm{i}\bar{\zeta}_1^2}{U}\frac{\mathrm{d}U}{\mathrm{d}t} \\
&= \frac{\mathrm{i}U}{2\zeta_1}(1 + \frac{1}{(2\mathrm{i}\cos\theta_1)^2} + \frac{\bar{\zeta}_1}{2\rho_1}\frac{1}{2\mathrm{i}\cos(\theta_1)}) - \frac{\mathrm{i}\bar{\zeta}_1^2}{U}\frac{\mathrm{d}U}{\mathrm{d}t} \\
&= \mathrm{i}\bar{\zeta}_1^2[\frac{U}{2\rho_1^2\bar{\zeta}_1}(1 - \frac{1}{4\cos^2\theta_1} - \frac{e^{\mathrm{i}\theta_1}}{4\cos\theta_1}) - \frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}t}] \\
&= \mathrm{i}\bar{\zeta}_1^2[\frac{U}{2\rho_1^3}(\mathrm{i}\cos\theta_1 + \sin\theta_1)(\frac{3}{4} - \frac{1}{4\cos^2\theta_1} - \frac{\mathrm{i}\tan\theta_1}{4}) - \frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}t}] \\
&= \mathrm{i}\bar{\zeta}_1^2[\frac{U}{2\rho_1^3}(\frac{3\sin\theta_1}{4} - \frac{\sin\theta_1}{4\cos^2\theta_1} + \frac{\sin\theta_1}{4} \\
&\qquad + \mathrm{i}(\frac{3\cos\theta_1}{4} - \frac{1}{4\cos\theta_1} - \frac{\sin^2\theta_1}{4\cos\theta_1})) - \frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}t}] \\
&= \mathrm{i}\bar{\zeta}_1^2[\frac{U}{2\rho_1^3}(\frac{\sin\theta_1(4\cos^2\theta_1 - 1)}{4\cos^2\theta_1} + \mathrm{i}\frac{\cos 2\theta_1}{2\cos\theta_1}) - \frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}t}].
\end{aligned}
$$

Divide both sides by $\mathrm{i}\bar{\zeta}_1^2$,

$$\frac{3\dot{\rho}_1}{\rho_1} + 2\mathrm{i}\theta_1 + \dot{\theta}_1\tan\theta_1 = \frac{U}{2\rho_1^3}(\frac{\sin\theta_1(4\cos^2\theta_1 - 1)}{4\cos^2\theta_1} + \mathrm{i}\frac{\cos 2\theta_1}{2\cos\theta_1}) - \frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}t}. \quad (69)$$

Collect the terms from the imaginary part,

$$2\dot{\theta}_1 = \frac{U}{4\rho_1^3}\frac{\cos 2\theta_1}{\cos\theta_1}.$$

Or

$$\frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \frac{U}{8\rho_1^3}\frac{\cos 2\theta_1}{\cos\theta_1}. \quad (70)$$

On the other hand, the real part gives

$$\frac{3\dot{\rho}_1}{\rho_1} + \dot{\theta}_1\tan\theta_1 = \frac{U}{2\rho_1^3}\frac{\sin\theta_1(4\cos^2\theta_1 - 1)}{4\cos^2\theta_1} - \frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}t}.$$

Use (70) to substitute $\dot{\theta}_1$,

$$\frac{3\dot{\rho}_1}{\rho_1} + \frac{U}{8\rho_1^3}\frac{\sin\theta_1\cos 2\theta_1}{\cos^2\theta_1} = \frac{U}{2\rho_1^3}\frac{\sin\theta_1(4\cos^2\theta_1 - 1)}{4\cos^2\theta_1} - \frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}t}.$$

Then we have

$$
\begin{aligned}
\frac{3\dot{\rho}_1}{\rho_1} &= \frac{U\sin\theta_1}{8\rho_1^3\cos^2\theta_1}(-\cos 2\theta_1 + 4\cos^2\theta_1 - 1) - \frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}t} \\
&= \frac{U\sin\theta_1}{8\rho_1^3\cos^2\theta_1}(2\cos^2\theta_1) - \frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}t} \\
&= \frac{U\sin\theta_1}{4\rho_1^3} - \frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}t}.
\end{aligned}
$$

Hence we obtain (11),

$$
\frac{\mathrm{d}\rho_1}{\mathrm{d}t} = \frac{U\sin(\theta_1)}{12\rho_1^2} - \frac{\rho_1}{3U}\frac{\mathrm{d}U}{\mathrm{d}t}. \tag{71}
$$

D Derivation of the dynamical system

We start with writing $\zeta = \xi + \mathrm{i}\eta$ and substitute this into (27). The left-hand side of (27) is

$$
\begin{aligned}
-\frac{3\mathrm{i}\eta^2\eta'}{\eta - \mathrm{i}\xi} + \frac{\eta\xi\eta'}{\eta - \mathrm{i}\xi} - \frac{\mathrm{i}\xi^2\eta'}{\eta - \mathrm{i}\xi} \\
- \frac{\xi^3\eta'}{\eta(\eta - \mathrm{i}\xi)} + \frac{2\eta^2\xi'}{\eta - \mathrm{i}\xi} \\
- \frac{2i\eta\xi\xi'}{\eta - \mathrm{i}\xi} + \frac{4\xi^2\xi'}{\eta - \mathrm{i}\xi}.
\end{aligned} \tag{72}
$$

Meanwhile the right-hand side of (27) is

$$
\frac{U\left(2\eta^2 - \mathrm{i}\eta\xi - \xi^2\right)}{8\eta^2(\eta - \mathrm{i}\xi)}. \tag{73}
$$

The real part of (72) can be written

$$
\xi\left(\eta^3 - \eta\xi^2\right)\eta' + 2\eta^2\left(\eta^2 + 2\xi^2\right)\xi'. \tag{74}
$$

The imaginary part of (72) is

$$
\eta^2\left(-\left(3\eta^2 + \xi^2\right)\eta' - 2\eta\xi\xi'.\right. \tag{75}
$$

On the other hand, the real part and the imaginary part of (73) are respectively,

$$\frac{1}{8}U\left(2\eta^2-\xi^2\right). \tag{76}$$

$$-\frac{1}{8}U\eta\xi. \tag{77}$$

By equating the real part and the imaginary part, we obtain the following two equations.

$$\xi\left(\eta^3-\eta\xi^2\right)\eta'+2\eta^2\left(\eta^2+2\xi^2\right)\xi'=\frac{1}{8}U\left(2\eta^2-\xi^2\right). \tag{78}$$

$$\eta^2\left(-\left(3\eta^2+\xi^2\right)\eta'-2\eta\xi\xi'\right)=-\frac{1}{8}U\eta\xi. \tag{79}$$

After simplifications, we finally have

$$\xi'=\frac{U\left(3\eta^2-\xi^2\right)}{24\left(\eta^2+\xi^2\right)^2}, \tag{80}$$

$$\eta'=-\frac{U\xi\left(\eta^2-3\xi^2\right)}{24\eta\left(\eta^2+\xi^2\right)^2}. \tag{81}$$

E The conformal mappings

E.1 Mapping (35)

We perform the following transformations.

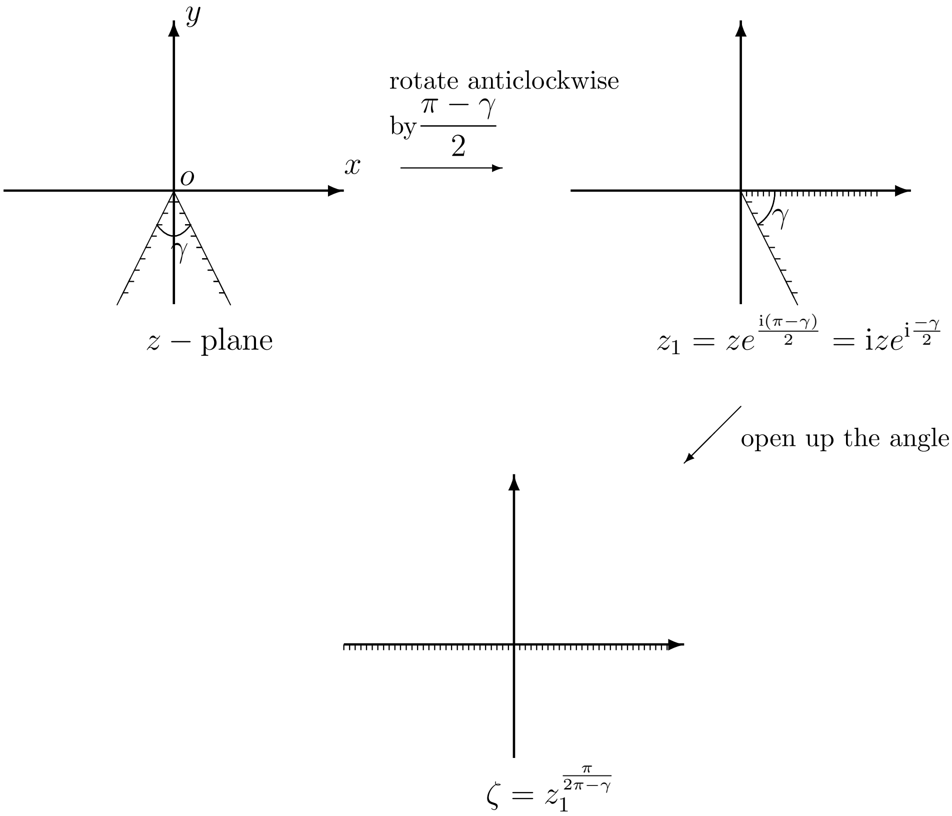

Figure 26: finite-angle wedge in the physical plane
and in the mapped plane

Hence the conformal mapping is defined as $z = -ie^{i\frac{\gamma}{2}}\zeta^{\frac{2\pi-\gamma}{\pi}}$.

E.2 Mapping (48)

One can easily check that $\zeta = z + \sqrt{z^2-1}$ maps the region in the z-plane onto the upper-half plane in the ζ-plane. Let $z^0 = x$ where $x \in (-1, 1)$. Then

$\zeta = x + \sqrt{x^2-1} = x + \mathrm{i}\sqrt{1-x^2}$. Obviously the image of the segment in the mapped plane is an upper-half unit circle. Hence the lower-half (upper-half) plane in the physical z-plane corresponds to the interior (exterior) of the half unit circle. Flows travel upwards in the physical z-plane. Hence in the mapped plane flows are emitted from the origin and go outwards the half-unit circle.

F Real part and imaginary part of (53)

Let $\zeta = \xi + \mathrm{i}\eta$ in (53). The real part gives

$$\begin{aligned}&\left(\left(\xi + (-2+\xi)\left(\eta^2+\xi^2\right)\right)\left(\xi\left(1-3\eta^4-2\xi^2+\xi^4-2\eta^2\left(1+\xi^2\right)\right)\eta'\right.\right.\\&+\eta\left(\left(1+\eta^2\right)^2-2\left(-1+\eta^2\right)\xi^2-3\xi^4\right)\xi'))/\left(2\eta\left(\eta^2+(-1+\xi)^2\right)\xi\left(\eta^2\right.\right.\\&\quad+\xi^2\left(\eta^2+(1+\xi)^2\right)) + \frac{-2\eta\xi\eta' + \left(\eta^4-\xi^2+\xi^4+\eta^2\left(1+2\xi^2\right)\right)\xi'}{2\left(\eta^2+\xi^2\right)^2} = \\&\quad-\frac{\xi\left(-1+(-2+\eta)\eta+\xi^2\right)\left(-1+\eta^2+\xi^2\right)\left(-1+\eta(2+\eta)+\xi^2\right)}{16\pi\eta^2\left(\eta^4+\left(-1+\xi^2\right)^2+2\eta^2\left(1+\xi^2\right)\right)}\end{aligned} \tag{82}$$

The imaginary part gives

$$\begin{aligned}&-\frac{\left(\eta^4-\xi^2+\xi^4+\eta^2\left(1+2\xi^2\right)\right)\eta'+2\eta\xi\xi'}{2\left(\eta^2+\xi^2\right)^2}+\\&\qquad\left(\left(-1+\eta^2+\xi^2\right)\left(\xi\left(1-3\eta^4-2\xi^2+\xi^4-2\eta^2\left(1+\xi^2\right)\right)\eta'+\right.\right.\\&\eta\left(\left(1+\eta^2\right)^2-2\left(-1+\eta^2\right)\xi^2-3\xi^4\right)\xi'))/\left(2\left(\eta^2+(-1+\xi)^2\right)\xi\left(\eta^2+\xi^2\right)\left(\eta^2+\right.\right.\\&\qquad(1+\xi)^2) = -\frac{\eta\left(1+\eta^2+\xi^2\right)\left(\left(1+\eta^2\right)^2+2\left(3+\eta^2\right)\xi^2+\xi^4\right)}{16\pi\xi^2\left(\eta^4+\left(-1+\xi^2\right)^2+2\eta^2\left(1+\xi^2\right)\right)}\end{aligned} \tag{83}$$

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