Extreme eigenvalue problem of LUE

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Abstract

In this paper, we investigate the extreme eigenvalue distribution function of a n-dimensional random matrix from Laguerre Matrices Ensemble in the unitary case(call it simply LUE later).

$$\text{namely } \mathbb{P}(\lambda_{max} \leq t).$$

We start by writing down the joint probability density whose weight function is Laguerre weight defined as

$$\omega(x) = x^{\alpha} e^{-x}.$$

where $\alpha > -1, x > 0$. Then we make a change of variable to transform the above weight into Jacobi time-dependent weight. With some classical work in random matrix theory, we relate the probability to a new variable H_n which is a solution of the continuous σ-form of P_V. In order to study the behaviour, as n$\rightarrow \infty$, of such probability, we try to not only look at the terms in P_V on the leading-order but also on the second-leading-order. For that purpose, we make at first a scaling

$$t = 4n - csn^{\frac{1}{3}}.$$

which is following from the equilibrium density in Laguerre's case, where s is a new variable that we will be working through the rest. Secondly, a further expansion is needed to be made here. We substitute these two back into the original P_V. After some rearrangements, we are able to collect two differential equations on the first two leading-orders. Further asymptotic analysis allows us to simplify the expression. We will see how this target can be achieved in section 4.

In the final section, we review the core result from Choup's paper. Several interesting remarks can be made.

Contents

1 Introduction

Many physical systems can be represented mathematically as problems in matrices. In the last century, physicists found that they were unable to analyse such models in some areas, such as microscopic environment, using modern mathematics. They did not give up but switched to use statistical science. Then random matrix theory was born as a new subject with numerous applications to engineering, finance etc. Now LUE is often applied to the problem of singular value decomposition(SVP).
As eigenvalue plays a significant role in matrix, one core branch of the theory is the extreme eigenvalue problem. People worked a lot on this topic in the last several decades. They have investigated perfectly the large behaviour (as n goes to infinity) of such problem for typical matrices ensembles. Tracy and Widom found the probability distribution of the largest eigenvalue of a random hermitian matrix in the edge scaling limit in 1993. The distribution is then named GUE Tracy-Widom distribution which we will see later in this paper. Entering the 21th century, mathematicians were not satisfied with the accuracy of this work. Thus they started to study more on the correction in the probability. In 2006, Choup had got an idea that is to use the Edgeworth expansion(1905) for the cases of GUE and LUE. (He also worked on GSE,GOE later in 2008) . One advantage of his approach is that the correction can appear automatically with its order. However, it is quite disappointing that Choup did not give an explicit expression at last.

In this paper, we want to study $\mathbb{P}(\lambda_{max} \leq t)$, the extreme eigenvalue problem of Laguerre Unitary Ensemble(LUE). Departing from the joint probability density of arbitrary eigenvalues in this case, we make a change of variable in the expression to let it be in time-dependent Jacobi form. With the help from [5], we do some classical work in orthogonal polynomials to derive a differential equation of Painlevé V. After that, the probability is related to a solution, H_n say, of the Painlevé equation of type V. It is not difficult to study the leading-order of the mentioned equation as n tends to infinity by using asymptotic analysis. Normally the study ends here but we want to work more on the correction. Thus the next step is to expand H_n to collect more information on the second-leading-order which is essential for determining the correction. Once this job is done, we can have two differential equations. Then making a further asymptotic analysis allows simple expressions. Then we substitute the asymptotic expansions back into the original problem. Favorable outcomes can be found and we will see these in section 4. Finally, it is interesting to compare our result to that of Choup. Remarkable comments are made for conclusion.

2 A link to Jacobi time-dependent weight

It is well known, see [16],[22] or [25], that the joint probability density function of arbitrary eigenvalues is

$$p(x_1, x_2..., x_n) = \frac{1}{D_n}\frac{1}{n!}[\Delta_n(x)]^2 \prod_{1\leq k\leq n} \omega(x_k).$$

where $x_i \in (a, b)$

Definition 2.1 *The Vandermonde Identity* $\Delta_n(x)$ *is given by*

$$\Delta_n(x) := \prod_{1\leq i<j\leq n} (x_j - x_i).$$

where $x = (x_1, x_2, ..., x_n)^T$.

Definition 2.2 D_n *is called the normalization constant and*

$$D_n = \frac{1}{n!}\int_{(a,b)^n} [\Delta_n(x)]^2 \prod_{1\leq k\leq n} \omega(x_k)dx_k.$$

ω is the weight function which is closely related to the orthogonal polynomials and the matrices ensemble. For example, if $\omega(x) = e^{-\frac{\beta}{2}x^2}$ we will get the Gaussian matrices ensemble. In our problem, we replace ω by the Laguerre weight.

Definition 2.3 *Laguerre weight is defined as*

$$\omega(x) = x^\alpha e^{-x}.$$

where $\alpha > -1, x > 0$.

Now we can write down the probability of the largest eigenvalue, that is smaller than a variable t, in a Laguerre's way.

Theorem 2.4

$$\begin{aligned}
&\mathbb{P}(\lambda_{max} \leq t) = \mathbb{P}(\text{all eigenvalues} \leq t) \\
&= \frac{\frac{1}{n!}\int_{(0,t)^n} [\Delta_n(x)]^2 \prod_{1\leq k\leq n} x_k^\alpha e^{-x_k} dx_k}{\frac{1}{n!}\int_{(0,\infty)^n} [\Delta_n(x)]^2 \prod_{1\leq k\leq n} x_k^\alpha e^{-x_k} dx_k} \\
&= \frac{1}{D_n}\frac{1}{n!}\int_{(0,t)^n} [\Delta_n(x)]^2 \prod_{1\leq k\leq n} x_k^\alpha e^{-x_k} dx_k. \qquad (2.1)
\end{aligned}$$

where D_n *is value of the denominator in the second line. It is called the normalization constant.*

Write

$$D_n(t) = \frac{1}{n!}\int_{(0,t)^n} [\Delta_n(x)]^2 \prod_{1\le k\le n} x_k^{\alpha} e^{-x_k} dx_k.$$

so $D_n = D_n(\infty)$. We have immediately that

$$\mathbb{P}(\lambda_{max} \le t) = \frac{D_n(t)}{D_n(\infty)}.$$

Remark 2.5 *For a Laguerre problem, x_k is only defined over $\mathbb{R}^+$. So in the integrals in theorem 2.4, the lower bounds are always zero.*

Now we perform a change of variable.

Let $x_k = ty_k$ **for** $\forall$**k ,** then

$$D_n(t) = \frac{1}{n!} t^{n(n+\alpha)} \int_{(0,1)^n} [\Delta_n(y)]^2 \prod_{1\le k\le n} y_k^{\alpha} e^{-ty_k} dy_k.$$

Definition 2.6 *We generalise D_n as*

$$D_{n,\beta}(t) = \frac{1}{n!} t^{n(n+\alpha)} \int_{(0,1)^n} [\Delta_n(y)]^2 \prod_{1\le k\le n} y_k^{\alpha}(1-y_k)^{\beta} e^{-ty_k} dy_k.$$

It is obvious that D_n is a special case of $D_{n,\beta}$ with $\beta = 0$. Now we will make another change of variable for $D_{n,\beta}$ to generate the Jacobi weight function.

Let $z_k = 2y_k - 1$ **for** $\forall$**k,** then $y_k = \frac{1+z_k}{2}$ and $1 - y_k = \frac{1-z_k}{2}$,

$$\Downarrow$$

$$D_{n,\beta} = \frac{t^{n(n+\alpha)}}{n!} 2^{-n(n+\alpha+\beta)} e^{-\frac{nt}{2}} \int_{(-1,1)^n} [\Delta_n(z)]^2 (1+z_k)^{\alpha} (1-z_k)^{\beta} e^{-\frac{tz_k}{2}} dz_k. \tag{2.2}$$

Definition 2.7 *Now we consider another quantity $\tilde{D}_n$ defined as*

$$\tilde{D}_{n,\beta}(t) = \frac{1}{n!}\int_{(-1,1)^n} [\Delta_n(z)]^2 (1+z_k)^{\alpha}(1-z_k)^{\beta} e^{-tz_k} dz_k. \tag{2.3}$$

The weight function in the previous integral is exactly the time-dependent Jacobi weight. This quantity is related to a solution to the Painlevé V. We will show this briefly in the next section.

Remark 2.8 *We have so far*

$$\mathbb{P}(\lambda_{max} \le t) = K(t)\tilde{D}_{n,0}(\frac{t}{2}).$$

where $K(t)$ is a known function shown above.

3 Jacobi polynomials and Painlevé V

People have already done similar work on this topic(see [5]). As this job is quite essential which cannot be skipped, we need to explain things in very details in this part. First of all, we introduce some notations and definitions. Let $\{P_j(x)\}$ be a sequence of orthogonal polynomials with respect to the Jacobi-weight function $\omega_{ja}(x) = (1+x)^{\alpha}(1-x)^{\beta}e^{-tx}$. We may write $\omega_{ja}(x) = \omega_0(x)e^{-tx}$. We also normalize the monic polynomials as

$$P_n(z) = z^n + \mathrm{p}_1(n,t)z^{n-1} + ...P_n(0).$$

3.1 Preliminaries

Definition 3.1 *The orthogonal condition is defined as*

$$\int_{-1}^{1} P_i(x)P_j(x)\omega_{ja}(x)dx = h_i(t)\delta_{i,j}.$$

Definition 3.2 α_n *and* β_n *are the coefficients defined as*

$$zP_n(z) = P_{n+1}(z) + \alpha_n P_n(z) + \beta_n P_{n-1}(z). \tag{3.1}$$

with $P_0(z) = 1, \beta_0 = 0.$

We obtain immediately that

$$\mathrm{p}_1(n,t) = -\sum_{j=0}^{n-1} \alpha_j(t). \tag{3.2}$$

since $\alpha_n(t) = \mathrm{p}_1(n,t) - \mathrm{p}_1(n+1,t)$.

Definition 3.3 *The moment can be written as*

$$\mu_i(t) = \int_{-1}^{1} x^i \omega_{ja}(x)dx.$$

Remark 3.4 $\tilde{D}_n(t)$ *can be evaluated as the Hankel determinant.*

$$i.e.\ \tilde{D}_n(t) = det(\mu_{i+j}(t))_{i,j=0}^{n-1} = \prod_{i=0}^{n-1} h_i(t).$$

Theorem 3.5 *Following from the definitions above, we can find immediately*

$$\frac{d}{dt}\ln \tilde{D}_n(t) = \mathrm{p}_1(n,t).$$

Proof We differentiate h_n with respect to t and get

$$\begin{aligned} h_n'(t) &= -\int_{-1}^{1} \omega_0(x) x e^{-tx} P_n^2(x) dx \\ &= -\int_{-1}^{1} \omega_0(x) e^{-tx} P_n(x)(xP_n(x)) dx \\ &= -\int_{-1}^{1} \omega_0(x) e^{-tx} P_n(x)(P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x)) dx \\ &= -\int_{-1}^{1} \omega_0(x) e^{-tx} \alpha_n P_n^2(x) dx = -\alpha_n h_n. \end{aligned}$$

We obtain

$$\frac{d}{dt} \ln(h_n(t)) = -\alpha_n. \tag{3.3}$$

from (3.2)&(3.3) we have

$$\frac{d}{dt} \ln \tilde{D}_n(t) = \sum_{i=0}^{n-1} \frac{d}{dt} \ln(h_i(t)) = -\sum_{i=0}^{n-1} \alpha_i = \boxed{\mathrm{p}_1(n,t)}.$$

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As can be seen from the previous pages, the coefficients α_n and β_n are closely related to the polynomials P_n. Then we can easily find the following differential system which can be used to derive the Painlevé V.

Theorem 3.6 (Toda equations) *$\alpha_n(t)$ and $\beta_n(t)$ defined in (3.1) satisfy a system of differential equations*

$$\begin{aligned} \beta_n' &= (\alpha_{n-1} - \alpha_n)\beta_n, \\ \alpha_n' &= \beta_n - \beta_{n+1.} \end{aligned} \tag{3.4}$$

Proof By definition,

$$\beta_n = \int_{-1}^{1} \omega_{ja}(x) P_{n-1}(x)(xP_n(x)) dx$$

$$\begin{aligned} \Rightarrow \beta_n h_{n-1} &= \int_{-1}^{1} \omega_{ja}(x)(\beta_n P_{n-1}) P_{n-1} dx \\ &= \int_{-1}^{1} \omega_{ja}(x)(xP_n - P_{n+1} - \alpha_n P_n) P_{n-1} dx. \end{aligned}$$

$$= \int_{-1}^{1} \omega_{ja}(x) x P_n P_{n-1} dx$$
$$= \int_{-1}^{1} \omega_{ja}(x) P_n (x P_{n-1}) dx$$
$$= \int_{-1}^{1} \omega_{ja}(x) P_n (P_n + \alpha_{n-1} P_{n-1} + \beta_{n-1} P_{n-2}) dx$$
$$= \int_{-1}^{1} \omega_{ja}(x) P_n^2(x) dx = h_n.$$

Hence $\beta_n = \frac{h_n}{h_{n-1}}$, now we take the derivative of β_n with respect to t

$$\beta_n' = \frac{h_n'}{h_{n-1}} - \frac{h_n h_{n-1}'}{h_{n-1}^2} = -\frac{\alpha_n h_n}{h_{n-1}} + \frac{\alpha_{n-1} h_{n-1} h_n}{h_{n-1}^2} = -\alpha_n \beta_n + \alpha_{n-1} \beta_n.$$

We have (the first toda equation)

$$\beta_n' = (\alpha_{n-1} - \alpha_n) \beta_n.$$

On the other hand, we consider

$$\int_{-1}^{1} \omega_{ja}(x) P_n(x) P_{n-1}(x) dx = 0, \ \forall t \in \mathbb{R}.$$

$$\Rightarrow \frac{d}{dt} \int_{-1}^{1} \omega_{ja}(x) P_n(x) P_{n-1}(x) dx = 0, \ \forall t \in \mathbb{R}.$$

$$\Leftrightarrow - \int_{-1}^{1} \omega_{ja}(x) P_n(x) (x P_{n-1}(x)) dx + \int_{-1}^{1} \omega_{ja}(x) P_n'(x) P_{n-1}(x) dx = 0.$$

$$\Leftrightarrow -h_n + \int_{-1}^{1} \omega_{ja}(x) \{x^n + \mathrm{p}_1'(n,t) x^{n-1} + ... P_n(0)\} P_{n-1}(x) dx = 0.$$

$$\Leftrightarrow -h_n + \mathrm{p}_1'(n,t) h_{n-1} = 0.$$

$$\Leftrightarrow \boxed{\mathrm{p}_1'(n,t) = \frac{h_n}{h_{n-1}} = \beta_n}.$$

$\alpha_n(t) = \mathrm{p}_1(n,t) - \mathrm{p}_1(n+1,t)$. We differentiate this equation with respect to t and obtain (the second Toda equation)

$$\alpha_n' = \mathrm{p}_1'(n,t) - \mathrm{p}_1'(n+1,t) = \beta_n - \beta_{n+1}.$$

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3.2 Ladder operators

Definition 3.7 *A potential v is defined by $\omega(x) = e^{-v(x)}$ where ω is a weight.*

The lowering and raising operators, see [6],[10], for polynomials which are orthogonal with respect to the Jacobi time-dependent weight are as the following

$$P_n'(z) = -B_n(z)P_n(z) + \beta_n A_n(z)P_{n-1}(z), \tag{L}$$

$$P_{n-1}'(z) = [B_n(z) + v'(x)]P_n(z) + A_{n-1}(z)P_n(z). \tag{R}$$

where A_n and B_n are two quantities shown as below

$$A_n(z) = \frac{1}{h_n}\int_{-1}^{1} \frac{v'(z) - v'(y)}{z - y} P_n^2(y)\omega_{ja}(y)dy,$$

$$B_n(z) = \frac{1}{h_{n-1}}\int_{-1}^{1} \frac{v'(z) - v'(y)}{z - y} P_{n-1}(y)P_n(y)\omega_{ja}(y)dy.$$

Two compatibility conditions (S_1), (S_2) are

$$B_{n+1}(z) + B_n(z) = (z - \alpha_n)A_n(z) - v'(z), \tag{S_1}$$

$$1 + (z - \alpha_n)(B_{n+1}(z) - B_n(z)) = \beta_{n+1}A_{n+1}(z) - \beta_n A_{n-1}(z). \tag{S_2}$$

Multiplying (S_2) by $A_n(z)$, we get (S_2')

$$B_n^2(z) + v'(z)B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \beta_n(z)A_n(z)A_{n-1}(z). \tag{S_2'}$$

The derivation of the above equations are quite straight. See [14] and [15]

Summary of what we have got so far:

$$\omega_{ja}(x) = (1 + x)^{\alpha}(1 - x)^{\beta}e^{-tx},$$

$$v(x) = -\alpha\ln(1 + x) - \beta\ln(1 - x) + tx,$$

$$v'(x) = -\frac{\alpha}{1 + x} - \frac{\beta}{1 - x} + t,$$

$$\frac{v'(z) - v'(y)}{z - y} = \frac{\alpha}{(y + 1)(z + 1)} + \frac{\beta}{(y - 1)(z - 1)}.$$

Now we introduce two new variables R_n and r_n which are related to a solution to Painlevé V.

Theorem 3.8

$$A_n(z) = -\frac{R_n(t)}{z-1} + \frac{t+R_n(t)}{z+1}, \qquad (E_1)$$

$$B_n(z) = -\frac{r_n(t)}{z-1} + \frac{r_n(t)-n}{z+1}. \qquad (E_2)$$

where

$$R_n(t) = \frac{\beta}{h_n}\int_{-1}^{1}\frac{P_n^2(y)}{1-y}(1-y)^\beta(1+y)^\alpha e^{-ty}dy,$$

$$r_n(t) = \frac{\beta}{h_{n-1}}\int_{-1}^{1}\frac{P_n(y)P_{n-1}(y)}{1-y}(1-y)^\beta(1+y)^\alpha e^{-ty}dy.$$

Proof Use the definitions of A_n and B_n then integrate by parts.

$$A_n(z) = \frac{1}{h_n}\int_{-1}^{1}(\frac{\alpha}{(y+1)(z+1)} + \frac{\beta}{(y-1)(z-1)})P_n^2(y)(1-y)^\beta(1+y)^\alpha e^{-ty}dy$$

$$= \frac{\alpha}{h_n(1+z)}\int_{-1}^{1}\frac{P_n^2}{1+y}(1-y)^\beta(1+y)e^{-ty}dy + \frac{1}{1-z}\frac{\beta}{h_n}\int_{-1}^{1}\frac{P_n^2}{1-y}(1-y)^\beta(1+y)^\alpha e^{-ty}dy$$

$$= \frac{1}{h_n(1+z)}\int_{-1}^{1}P_n^2(1-y)^\beta\alpha(1+y)^{\alpha-1}e^{-ty}dy + \frac{R_n(t)}{1-z}$$

$$= \frac{1}{h_n(1+z)}\int_{-1}^{1}P_n^2(1-y)^\beta e^{-ty}d(1+y)^\alpha + \frac{R_n(t)}{1-z}.$$

Integrating by parts, we can get

$$A_n(z) = \frac{1}{z+1}\frac{1}{h_n}(h_nR_n(t)+th_n) + \frac{R_n(t)}{1-z} = \frac{t+R_n(t)}{z+1} + \frac{R_n(t)}{1-z}.$$

Similarly, we have

$$B_n(z) = -\frac{r_n(t)}{z-1} + \frac{r_n(t)-n}{z+1}.$$

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The following step is to find their properties which can lead us to the differential equation.

Theorem 3.9

$$-r_{n+1} - r_n = \beta - R_n(1-\alpha_n), \qquad (3.5)$$

$$r_{n+1} + r_n = (2n+1) + \alpha - (1+\alpha_n)(R_n(t)+t). \qquad (3.6)$$

Proof We substitute (E_2) into (S_1) and collect the coefficients of $\frac{1}{z-1}$.
It gives

$$-R_n(t)(1-\alpha_n) = -(r_{n+1} + r_n + \beta).$$

which is exactly (3.5).

On the other hand, we may also look at the coefficient of $\frac{1}{z+1}$,

$$r_{n+1} + r_n - (2n+1) - \alpha = -(1+\alpha_n)(R_n(t) + t).$$

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Theorem 3.10

$$r_n^2 + \beta\alpha_n = \beta_n R_n R_{n-1}, \tag{3.7}$$

$$(r_n - n)^2 - \alpha(r_n - n) = \alpha_n(R_n + t)(R_{n-1} + t), \tag{3.8}$$

$$\frac{\alpha-\beta}{2} r_n + \frac{\beta n}{2} - t r_n - r_n(r_n - n) - \sum_{0 \le j \le n-1} R_j = -\frac{\beta_n}{2}[R_n(R_{n-1}+t) + (R_n+t)R_{n-1}]. \tag{3.9}$$

Proof We substitute (E_1) and (E_2) into (S_2').

The coefficient of $\frac{1}{(z-1)^2}$:

$$r_n^2 + \beta r_n = \beta_n R_n R_{n-1}.$$

The coefficient of $\frac{1}{(z+1)^2}$:

$$(r_n - n)^2 - \alpha(r_n - n) = \beta_n(t + R_n)(t + R_{n-1}).$$

The coefficient of $\frac{1}{(z-1)(z+1)}$:

$$-2r_n(r_n - n) + (\alpha - \beta)r_n + \beta n - 2t r_n - \sum_{0 \le j \le n-1} R_j$$

$$=\text{-}\beta_n[(t + R_n)R_{n-1} + (t + R_{n-1})R_n].$$

They are exactly the three equations in the theorem.

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(3.5)+(3.6) gives

$$2R_n = 2n + \alpha + \beta + 1 - t - t\alpha_n. \tag{3.10}$$

(3.5)-(3.6) and eliminating $\beta_n R_n R_{n-1}$ from (3.3) and (3.4) gives

$$n(n+\alpha) - (2n + \alpha + \beta)r_n = \beta_n(t^2 + t(R_n + R_{n-1})). \tag{3.11}$$

3.3 Riccati equations and P_V

In this short section, we will determine the Painlevé differential equation by introducing the following theorem.

Theorem 3.11 (*Riccati equations)We have a system of differential equations in* R_n *and* r_n *with respect to* t *as below*

$$t\frac{dR_n}{dt} = \beta t + (2n+1+\alpha+\beta-2t)R_n - 2R_n^2 + 2tr_n. \tag{3.12}$$

$$\frac{dr_n}{dt} = -\frac{R_n}{t(t+R_n)}[n(n+\alpha)+(2n+\alpha+\beta)r_n+\frac{t}{R_n}(r_n^2+\beta r_n)]+\frac{r_n^2+\beta r_n}{R_n}. \tag{3.13}$$

Proof We sum up (3.10) form 0 to $n-1$ and get

$$\sum_{j=0}^{n-1} R_j = \frac{n+\alpha+\beta-t}{2} - \frac{t}{2}\sum_{j=0}^{n-1}\alpha_j. \tag{3.14}$$

On the other hand, (3.9) gives

$$\begin{aligned}\sum_{0\le j\le n-1} R_j &= \frac{\alpha-\beta}{2}r_n+\frac{\beta n}{2}-tr_n-r_n(r_n-n)+\frac{\beta_n}{2}[R_n(R_{n-1}+t)+(R_n+t)R_{n-1}]\\ [\text{from (3.6)}] &= (\frac{\alpha+\beta}{2}+n-t)r_n+\frac{\alpha n}{2}+\frac{\beta_n t}{2}(R_n+R_{n-1})\\ [\text{from (3.7)}] &= \frac{n(n+\alpha+\beta)}{2} - tr_n - \frac{\beta_n t^2}{2}. \end{aligned}\tag{3.15}$$

Then equate (3.14) and (3.15), it returns

$$\frac{n+\alpha+\beta-t}{2} - \frac{t}{2}\sum_{j=0}^{n-1}\alpha_j = \frac{n(n+\alpha+\beta)}{2} - tr_n - \frac{\beta_n t^2}{2}$$

$$\Leftrightarrow \mathrm{p}_1(n,t) = -\sum_{j=0}^{n-1}\alpha_j = n - 2r_n - t\beta_n. \tag{3.16}$$

So

$$\begin{aligned}\mathrm{p}_1(n,t)-\mathrm{p}_1(n+1,t) = \alpha_n &= -1+2(r_{n+1}-r_n)+t(\beta_{n+1}-\beta_n)\\ &= -1+2(r_{n+1}-r_n)+t\frac{d}{dt}(\mathrm{p}_1(n+1,t)-\mathrm{p}_1(n,t))\\ &= -1+2(r_{n+1}-r_n)-t\frac{d\alpha_n}{dt}.\end{aligned}$$

Hence

$$\alpha_n = 2(r_{n+1} - r_n) - 1 - t\frac{d\alpha_n}{dt}.$$

We eliminate r_{n+1} in the above equation using (3.6),

$$\frac{d}{dt}(t\alpha_n) = 2n + \alpha - \beta - t - (t + 2R_n)\alpha_n - 4r_n.$$

from(3.10) we are finally left with a differential equation in R_n

$$t\frac{d}{dt}R_n = \beta t + (2n + 1 + \alpha + \beta - 2t)R_n - 2R_n^2 + 2tr_n.$$

which is exactly (3.12). For r_n, we start with the two equations below

$$-\sum_{j=0}^{n-1}\alpha_j(t) = \mathrm{p}_1(n,t),$$

$$\mathrm{p}_1'(n,t) = \beta_n(t).$$

Now differentiate (3.16) with respect to t, one can observe

$$\beta_n = \mathrm{p}_1'(n,t) = -2\frac{dr_n}{dt} - \beta_n - t\frac{d\beta_n}{dt}.$$

Use the first toda equation to eliminate β_n',

$$\frac{dr_n}{dt} = -\beta_n - \frac{t}{2}(\alpha_{n-1} - \alpha_n)\beta_n.$$

Replace α_n by R_n with the help of (3.10),

$$\frac{dr_n}{dt} = -\beta_n - \beta_n(R_n - R_{n-1} - 1) = -\beta_n(R_n - R_{n-1})$$

$$= -\beta_n R_n + \frac{r_n^2 + \beta r_n}{R_n}.$$

where we have used (3.7) to express R_{n-1} in terms of R_n and α_n. And again (3.11) can throw away β_n and leave us a differential equation only in R_n and r_n, one can finally get

$$\frac{dr_n}{dt} = -\frac{R_n}{t(t+R_n)}\{n(n+\alpha) - (2n+\alpha+\beta)r_n - \frac{t}{R_n}(r_n^2 + \alpha r_n)\} + \frac{r_n^2 + \beta r_n}{R_n}.$$

as expected

■

Remark 3.12 *The work in section 3 is based on $\alpha, \beta > 0$. People have done advanced analysis to demonstrate that the result can be extended to the case where $\alpha, \beta > -1$. Our problem is located at $\beta = 0$. Hence it is fine to use directly the result.*

We have the coupled Riccati equations in hand. In order to identify with Painlevé V, one need to combine the two equations and switch off one variable to get a second order o.d.e only in R_n(or r_n). So take a derivative of (3.12) with respect to t and then replace (r_n,r_n') by (R_n,R_n') using (3.12)&(3.13). As the computation is really complicated, we do not give the details. One can manipulate with aid from mathematica or maple to carry out the operation mentioned above and plug the following change of variable directly into the equation.

$$Y(t) = 1 + \frac{t}{2}\frac{1}{R_n(\frac{t}{2})}. \tag{3.17}$$

Finally, The equation is converted into $P_V(\frac{\beta^2}{2}, -\frac{\alpha^2}{2}, 2n+1+\alpha+\beta, -\frac{1}{2})$.

3.4 Jimbo-Miwa σ-form of P_V

In order to link the probability defined in section 1 and the solution to the P_V, we need now to discover the JM σ-form of P_V. Look at (3.16) again.

$$\mathrm{p}_1(n,t) = n - 2r_n - t\beta_n = n - 2r_n - t\mathrm{p}_1'(n,t).$$

or

$$r_n(t) = \frac{1}{2}[n - \frac{d}{dt}(t\mathrm{p}_1(n,t))].$$

Substitute (3.17) back into (3.12), we compare the coefficients (see [23] or section 5 in [5]) and find the σ-function H_n

$$\frac{dH_n(t)}{dt} = -r_n(\frac{t}{2}) = -\frac{1}{2}[n - \frac{d}{dt}\mathrm{p}_1(n, \frac{t}{2})].$$

Then integrate and fix a constant,

$$H_n(t) = \frac{t}{2}\mathrm{p}_1(n, \frac{t}{2}) - \frac{nt}{2} + n(n+\alpha). \tag{3.18}$$

And the σ-form of P_V is written as

$$(tH_n'')^2 = [H_n + (2n+\alpha+\beta-t)H_n']^2 + 4[H_n - n(n+\alpha) - tH_n'](H_n'^2 - \beta H_n'). \tag{†}$$

with initial conditions $H_n(0) = n(n+\alpha)$, $H_n'(0) = -\frac{n(n+\beta)}{\alpha+\beta+2n}$.

Comment: We know the probability $\mathbb{P}$ can be written as

$$\mathbb{P}(\lambda_{max} \le t) = K(t)\tilde{D}_{n,0}(\frac{t}{2}).$$

where $K(t)$ is known, $\tilde{D}_{n,0}$ is defined previously in Definition 2.7. Then we prove that $\tilde{D}_{n,0}$ is closely related to H_n in section 4.

4 The correction term

In this part, we would like to investigate the large behaviour of the probability. The first step is to study the leading-order. Afterwards we can do more on the correction.

Recall

$$\mathbb{P}(\lambda_{max} \leq t) = \frac{1}{D_n}\frac{1}{n!}\int_{(0,t)^n}[\Delta_n(x)]^2\prod_{1\leq k\leq n} x_k^{\alpha}e^{-x_k}dx_k$$

$$= \frac{1}{D_n}\frac{1}{n!}t^{n(n+\alpha)}\int_{(0,1)^n}[\Delta_n(y)]^2\prod_{1\leq k\leq n} y_k^{\alpha}e^{-ty_k}dy_k$$

$$= \frac{1}{D_n}\frac{1}{n!}t^{n(n+\alpha)}\int_{(0,1)^n}[\Delta_n(y)]^2\prod_{1\leq k\leq n} y_k^{\alpha}(1-y_k)^{\beta}e^{-ty_k}dy_k$$

$$= \frac{1}{D_n}\frac{t^{n(n+\alpha)}}{n!}2^{-n(n+\alpha+\beta)}e^{-\frac{nt}{2}}\int_{(-1,1)^n}[\Delta_n(z)]^2(1+z_k)^{\alpha}(1-z_k)^{\beta}e^{-\frac{tz_k}{2}}dz_k$$

$$= \frac{1}{D_n}\frac{t^{n(n+\alpha)}}{n!}2^{-n(n+\alpha+\beta)}e^{-\frac{nt}{2}}\tilde{D}_n(\frac{t}{2}). \tag{4.1}$$

Theorem 4.1

$$t\frac{d}{dt}\ln\mathbb{P}(\lambda_{max} \leq t) = H_n(t).$$

Proof Take a logarithm of (4.1)

$$\ln\mathbb{P}(\lambda_{max} \leq t) = \text{constant} + n(n+\alpha)\ln(t) - \frac{nt}{2} + \ln\tilde{D}_n(\frac{t}{2}).$$

Take a further derivative with respect to t of the above equation,

$$\frac{d}{dt}\ln\mathbb{P}(\lambda_{max} \leq t) = \frac{n(n+\alpha)}{t} - \frac{n}{2} + \frac{d}{dt}\ln\tilde{D}_n(\frac{t}{2}). \tag{4.2}$$

Theorem 3.5 tells us

$$\frac{d}{dt}\ln\tilde{D}_n(t) = \mathrm{p}_1(n,t).$$

Thus

$$\frac{d}{dt}\ln\tilde{D}_n(\frac{t}{2}) = \frac{1}{2}\frac{d}{d\frac{t}{2}}\ln\tilde{D}_n(\frac{t}{2}) = \frac{1}{2}\mathrm{p}_1(n,\frac{t}{2}). \tag{4.3}$$

Plug (4.3) into (4.2),

$$\frac{d}{dt}\ln\mathbb{P}(\lambda_{max} \leq t) = \frac{n(n+\alpha)}{t} - \frac{n}{2} + \frac{1}{2}\mathrm{p}_1(n,\frac{t}{2}).$$

Multiplying by t,

$$t\frac{d}{dt}\ln\mathbb{P}(\lambda_{max} \leq t) = n(n+\alpha) - \frac{nt}{2} + \frac{t}{2}\mathrm{p}_1(n,\frac{t}{2}).$$

Then identify with $H_n(t)$, it can be easily found

$$t\frac{d}{dt}\ln \mathbb{P}(\lambda_{max} \le t) = H_n.$$

■

Now we review the equation (†) and replace β by zero,

$$(tH_n'')^2 = [H_n + (2n+\alpha-t)H_n']^2 + 4[H_n - n(n+\alpha) - tH_n'](H_n'^2). \qquad (‡)$$

4.1 Equilibrium density

We consider the case where n is large.

Definition 4.2 *Let J be the support. The equilibrium density ρ may be treated as a fluid defined as $\rho(x)dx$, $x \in J$, then*

$$\forall I \subseteq J, \quad \int_I \rho(x)dx = \#(eigenvalues\ in\ I).$$

Proposition 4.3 *ρ is characterised by the following condition*

$$v(x) - 2\int_J \ln|x-y|\rho(y)dy = \lambda, \ x \in J. \qquad (4.4)$$

where v is a potential and λ is the Lagrange Multiplier determined by the condition of the total number of eigenvalues. i.e.

$$\int_J \rho(x)dx = n.$$

The proof of this proposition can be found in [26](prop3.4). Now we take a derivative of (4.4) with respect to t. It allows us to write a singular integral equation(see [28]),

$$2\mathrm{P}\int_J \frac{\rho(y)dx}{x-y} = v'(x), \ x \in J.$$

For us, $J = [0,b]$. We find the solution that vanishes at 0 and b by using the inversion formula for $x \in [0,b]$, (see [6])

$$\begin{aligned}
\rho(x) &= \frac{\sqrt{(b-x)(x-0)}}{2\pi^2}\int_0^b \frac{v'(x)-v'(y)}{x-y}\frac{dy}{\sqrt{(b-y)(y-0)}} \\
&= \frac{\sqrt{(b-x)x}}{2\pi^2}\int_0^b \frac{\alpha}{xy}\frac{dy}{\sqrt{(b-y)y}} \\
&= \frac{\alpha\sqrt{(b-x)x}}{2\pi^2 x}\int_0^b \frac{1}{y}\frac{dy}{\sqrt{(b-y)y}}
\end{aligned}$$

$$= \frac{\sqrt{(b-x)x}}{2\pi x}.(Mar\check{c}enko - Pastur\ 1967)$$

The finite condition is

$$\frac{1}{2\pi}\int_0^b \sqrt{\frac{(b-x)}{x}}dx = n. \qquad (\star)$$

Let us compute the integral on the left hand side. Before that, we do some preparatory work.

Recall (Beta function): Given that $a, b \in \mathbb{R}$, the Beta function is defined as

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt.$$

The most famous property of B which can be proved by using the polar coordinate is

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Now we can retake $(\star)$ and use the property of the Beta function. Let $x = by$. It can be easily found that

$$\begin{aligned} n &= \frac{b}{2\pi}\int_0^1 \sqrt{\frac{1-y}{y}}dy \\ &= \frac{b}{2\pi}B(\frac{1}{2},\frac{3}{2}) \\ &= \frac{b}{2\pi}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(2)} \\ &= \frac{b}{2\pi}\frac{\pi}{2} = \frac{b}{4}. \end{aligned}$$

So

$$b = 4n. \qquad (\diamond)$$

Then we think of the largest eigenvalue on the tail,

$$\mathrm{O}(1) = \frac{1}{2\pi}\int_t^b \sqrt{\frac{(b-x)}{x}}dx.$$

as $t \to b$,

$$\begin{aligned} &\sim \frac{1}{2\pi}\frac{1}{\sqrt{b}}\int_t^b \sqrt{b-x}dx \\ &= \frac{1}{2\pi\sqrt{b}}\int_0^{b-t}\sqrt{x}dx = \frac{1}{2\pi\sqrt{b}}\frac{2}{3}(b-t)^{\frac{3}{2}} \\ &\sim \frac{1}{3\pi}\frac{1}{\sqrt{4n}}(4n-t)^{\frac{3}{2}} = \frac{1}{6\pi\sqrt{n}}(4n-t)^{\frac{3}{2}}. \end{aligned}$$

$\Rightarrow (4n-t)^{\frac{3}{2}} = \mathrm{O}(n^{\frac{1}{2}})$, thus we can introduce a new variable s

$$t = 4n - csn^{\frac{1}{3}}.$$

where c is a constant. $c = 2^{\frac{4}{3}}$

4.2 Study on the leading-order

From section 4.1, a scaling can be made here

$$t = 4n - csn^{\frac{1}{3}}. \qquad (*)$$

With some calculations, we derive a differential equation of H_n in variable s. We write $H_n(4n-csn^{\frac{1}{3}}, n)$ as H_n for short. To study the leading-order, we need to prove a theorem first as below.

Theorem 4.4 *As $n \to \infty$,*

$$H_n = O(n^{-\frac{2}{3}}).$$

Proof We may start with writing $\sigma_n = n^{\xi} H_n = O(1)$ as $n \to \infty$.

- When $\xi < -\frac{2}{3}$,

Then the leading order of the differential equation gives

$$\frac{16}{c^3}\sigma_n'^3 = 0 \Rightarrow \sigma_n' = 0$$

With some boundary conditions, we have $\sigma_n = 0$. i.e. $\xi \geq -\frac{2}{3}$.

- When $\xi = -\frac{2}{3}$,

we have an equation obtained from the leading order

$$\frac{16}{c^2}\sigma_n''^2 = 4c\sigma_n\sigma_n' - 4cs\sigma_n + 16\sigma_n'^3$$

It shows that σ_n is not a zero-function.
So $\xi = -\frac{2}{3}$

■

Then we try to define a new variable σ_n as the following.

$$\sigma_n = \frac{H_n}{c^2 n^{\frac{2}{3}}}. \qquad (4.5)$$

Call $\sigma = \lim_{n\to\infty} \sigma_n$ Then we plug $(*)$,(4.5) into (‡) and collect the terms of leading order. Finally we can get

$$16\sigma''^2 = 4c^3\sigma\sigma' - 4c^3 s\sigma'^2 + 16c^3\sigma'^3.$$

Or

$$\sigma''^2 = 4\sigma\sigma' - 4s\sigma'^2 + 16\sigma'^3. \qquad (4.6)$$

where we replace c^3 by 16. Here σ' denotes $\frac{d}{ds}\sigma$. See the full equation in the appendix.

Remark 4.5 *The function σ has no business with α. i.e. α does not involve in the probability for large n.*

4.3 Further expansion of σ_n

We want to express not only the leading order term of σ_n but the correction term as well. Thus we need a further expansion of σ_n which may give us more information.

4.3.1 Possible values of γ

We may write σ_n as

$$\sigma_n(s) = \sigma(s) + n^{\gamma}\delta(s) + o(n^{\gamma}). \tag{4.7}$$

where γ is a negative value to be determined. Now we consider (4.5) (4.7) and (†) together. This time we can rearrange the terms by using mathematica (See the full results in the appendix) and give a list of orders, that are $\geq \frac{4}{3}$, for the both sides.

On the LHS: 2, $2{+}\gamma, 2+2\gamma, \frac{4}{3}$.

On the RHS: 2, $2{+}\gamma, 2+2\gamma, 2+3\gamma, \frac{5}{3}, \frac{5}{3}+\gamma, \frac{4}{3}$.

Order 2 represents the leading-order which we have already investigated in the latest part. One can observe that there is only one term, $-8c^2\alpha\sigma'^2$ say, whose order is $n^{\frac{5}{3}}$. It indicates that we must have at least one another term of $O(n^{\frac{5}{3}})$, otherwise σ' is a zero-function. Thus we have three possible values of γ. (4.7) allows us to collect the terms on the second-leading-order to form another differential equation.

$$\boxed{\frac{5}{3} = \begin{cases} 2+\gamma \\ 2+2\gamma \\ 2+3\gamma \end{cases}.}$$

■

In the next subsection, we want to eliminate two possible values of γ. The technique of achieving this goal is to consider the differential equation on the second-leading-order. Once this job is done, we are able to determine the exact value for γ.

4.3.2 $\gamma = -\frac{1}{9}$ or $-\frac{1}{6}$

i.e. $2 + 3\gamma = \frac{5}{3}$ or $2 + 2\gamma = \frac{5}{3}$

In these two cases, the second-leading-order is $O(n^{2+\gamma})$. We focus on the differential equation of this order and can get
At $O(n^{2+\gamma})$.

$$32\sigma''\delta'' = 4c^3\sigma'\delta + 4c^3\sigma\delta' - 8c^3 s\sigma'\delta' + 48c^3\sigma'^2\delta'.$$

Or

$$\sigma''\delta'' = 2\sigma'\delta + 2\sigma\delta' - 4s\sigma'\delta' + 24\sigma'^2\delta'. \tag{4.8}$$

We observe that (4.8) has no term in α and this implies that δ does not depend on α. It is necessary to stop here to make a summary.

Summary of this part :

1. δ and σ do not depend on α.

2. $\sigma_n(s) = \sigma(s) + n^{\gamma}\delta(s) + o(n^{\gamma})$.

3. $\sigma_n = \frac{H_n}{c^2 n^{\frac{2}{3}}}$, where H_n has a link with the original probability.

When n is fairly large but not quite, the probability varies little as α changes its value since δ and σ are functions only in s. People can prove this result by numerical methods(matlab is helpful to generate the random matrices of LUE). Overall, it is impossible to let γ be such values because α still plays an important role in $\mathbb{P}$.

Conclusion of this section: $\boxed{\gamma \text{ cannot be either } -\frac{1}{9} \text{ or } -\frac{1}{6}.}$

The last and only possible value for γ is $-\frac{1}{3}$. We will study this case carefully in the next section.

Remark 4.6 *As $n \to \infty$, $\sigma_n \to \sigma$ which does not depend on α. Thus the probability $\mathbb{P}$ varies little with α as n is sufficiently large.*

4.4 Case where $\gamma = -\frac{1}{3}$

. From earlier, we have derived that $\sigma_n(s) = \sigma(s) + n^{-\frac{1}{3}}\delta(s) + o(n^{-\frac{1}{3}})$. We want to express δ in function of σ. We collect the terms of first two leading-orders. It gives us two differential equations.
At $O(n^2)$(leading-order):

$$\sigma''^2 = 4\sigma\sigma' - 4s\sigma'^2 + 16\sigma'^3. \tag{4.9}$$

At $O(n^{\frac{5}{3}})$(second-leading-order):

$$\sigma''\delta'' = 2\sigma'\delta + 2\sigma\delta' - 4s\sigma'\delta' + 24\sigma'^2\delta' - \frac{c^2}{4}\alpha\sigma'^2. \tag{4.10}$$

As $s \to -\infty$, we do an asymptotic analysis on (4.9) and get

$$\sigma(s) = \frac{1}{16}s^2 - \frac{1}{32}s^{-1} + o(s^{-1}). \tag{4.11}$$

As $s \to -\infty$, (4.10) becomes

$$\delta'' - (2s + \frac{1}{2s^2})\delta = -2c^2\alpha(\frac{s^2}{64} + \frac{1}{128s}). \tag{4.12}$$

Remark 4.7 *In (4.12) we have neglected the term in δ' since the coefficient is of order $(\frac{1}{s^4})$. It will converge rapidly to zeros as $s \to -\infty$.*

Write

$$\delta(s) = \delta_P(s) + \delta_G(s).$$

where δ_P and δ_G are respectively the particular solution and the general solution. Now we do an asymptotic analysis again to get the particular solution,

$$\delta_P(s) = \frac{\alpha c^2}{64}(s - \frac{3}{2s}) + o(s^{-1}). \tag{4.13}$$

Then look at the homogenous equation of (4.12).

$$\delta'' - (2s + \frac{1}{2s^2})\delta = 0.$$

Notice that $\frac{1}{2s^2}$ is much smaller than $2s$ as s tends to -∞. Asymptotically, the homogenous equation can be written as below

$$\delta'' - 2s\delta = 0.$$

This equation is known as Airy's equation. Its solution is a sum of the two Airy's functions Ai and Bi. Thus the homogenous solution can be written as

$$\delta_H(s) = a_1 Ai(2^{\frac{1}{3}}s) + a_2 Bi(2^{\frac{1}{3}}s). \tag{4.14}$$

where a_1, a_2 are constants(it is tricky to determine a_1, a_2). For the limit in the negative direction, we have the following asymptotic formulae.

$$Ai(s) \sim \frac{\sin(-\frac{2}{3}s^{\frac{3}{2}} + \frac{\pi}{4})}{\sqrt{\pi}(-s)^{\frac{1}{4}}}.$$

$$Bi(s) \sim \frac{\cos(-\frac{2}{3}s^{\frac{3}{2}} + \frac{\pi}{4})}{\sqrt{\pi}(-s)^{\frac{1}{4}}}.$$

Hence

$$Ai(2^{\frac{1}{3}}s) \sim \frac{\sin(-\frac{2}{3}\sqrt{2}s^{\frac{3}{2}} + \frac{\pi}{4})}{\sqrt{\pi}2^{\frac{1}{12}}(-s)^{\frac{1}{4}}},$$
$$Bi(2^{\frac{1}{3}}s) \sim \frac{\cos(-\frac{2}{3}\sqrt{2}s^{\frac{3}{2}} + \frac{\pi}{4})}{\sqrt{\pi}2^{\frac{1}{12}}(-s)^{\frac{1}{4}}}.$$

Now we have the asymptotic expansions for σ and δ as s tends to minus infinity. The next step is to substitute these two back into the probability expression (shown in the theorem 4.1). Then we can see how the correction term generate in the next subsection.

4.5 Derivation of the correction term

We know from earlier

$$t\frac{d}{dt}\ln \mathbb{P}(\lambda_{max} \leq t) = t\frac{d}{dt}\ln F_{n,2}(t) = H_n.$$

Replace t by $4n - csn^{\frac{1}{3}}$ and H_n by $c^2 n^{\frac{2}{3}}\sigma_n$.

$$\begin{aligned}
\frac{d}{ds}\ln \mathbb{P}(n, 4n - csn^{\frac{1}{3}}) &= \frac{-c^3 n}{4n - csn^{\frac{1}{3}}}\sigma_n \\
&= -\frac{4}{1 - \frac{cs}{4n^{\frac{2}{3}}}}[\sigma + n^{-\frac{1}{3}}\delta + o(n^{-\frac{1}{3}})] \\
&= -4[1 - \frac{cs}{4n^{\frac{2}{3}}} + o(n^{-\frac{2}{3}})][(\sigma + n^{-\frac{1}{3}}\delta + o(n^{-\frac{1}{3}})] \\
&= -4(\sigma + n^{-\frac{1}{3}}\delta) + o(n^{-\frac{1}{3}}).
\end{aligned}$$

Thus

$$\begin{aligned}
F_{n,2}(s) = \mathbb{P}(n, 4n - csn^{\frac{1}{3}}) &= \exp[\int^s -4\sigma(u)du + n^{-\frac{1}{3}}\int^s -4\delta(u)du + o(n^{-\frac{1}{3}})] \\
&= \exp[\int^s -4\sigma(u)du]\exp[n^{-\frac{1}{3}}\int^s -4\delta(u)du]\exp[o(n^{-\frac{1}{3}})].
\end{aligned}$$

as $n \to \infty$,

$$\begin{aligned}
F_{n,2}(s) &= \exp[-\int^s 4\sigma(u)du][1 - n^{-\frac{1}{3}}\int^s 4\delta(u)du + o(n^{\frac{1}{3}})][1 + o(n^{\frac{1}{3}})] \\
&= \exp[-\int^s 4\sigma(u)du][1 - n^{-\frac{1}{3}}\int^s 4\delta(u)du] + o(n^{\frac{1}{3}}).
\end{aligned}$$

Now we integrate (4.11). After some simple computations, we can get

$$\int^s \sigma(u)du \sim [-\frac{1}{48}s^3 + \frac{1}{32}\ln(-s)] + \text{constant}_1.$$

Then take the exponential function of the above expression,

$$\exp[-4\int^{s}\sigma(u)du]\sim C_1(-s)^{-\frac{1}{8}}\exp(\frac{1}{12}s^3)=F_2(s). \qquad (\bigstar)$$

The constant $C_1=2^{\frac{1}{12}}e^{\zeta'(-1)}$(Torsten Ehrhardt; P. Deift, A. Its and I. Krasovky, circa 2006-2007).
On the other hand, we integrate (4.13) and (4.14)

$$\Delta_H(s)=\int^{s}\delta_H(u)du\sim\frac{\alpha c^2}{64}[\frac{s^2}{2}+\frac{3}{2}\ln(-s)]+\text{constant}_2,$$
$$\Delta_G(s)=\int^{s}\delta_G(u)du=O(n^{\frac{3}{4}}).$$

Call $\Delta(s)=\int^{s}\delta(u)du$. Obviously, we have $\Delta(s)=\Delta_G(s)+\Delta_H(s)$.
Finally, we obtain

$$F_{n,2}(s)=F_2(s)\{1+u(s)n^{-\frac{1}{3}}\}+o(n^{-\frac{1}{3}}), \qquad (4.15)$$

with

$$F_2(s)=C_1(-s)^{-\frac{1}{8}}\exp(\frac{1}{12}s^3),$$
$$u(s)=-4\Delta(s).$$

Note that $F_2(s)$ is exactly the Tracy- Widom cumulative distribution function. There are several constants to be determined for $\Delta(s)$. In $\Delta_H(s)$, we need to derive $a1, a2$ that are the constants of the Airy's functions. And then we have constant$_2$ in $\Delta_P(s)$ to be calculated. However, both jobs are really tricky. I am unable to solve this at the moment and leave it as an open problem.

Remark 4.8 *One can generalise the unitary case($\beta=2$) to $\beta-$ Laguerre ensemble with the same procedure as in the section 2,3,4. Finally, it can be found that $F_2(s)$ is replaced by $F_\beta(s)$, β-Tracy-Widom cumulative distribution function.*

Summary

$$F_2(s)=C_1(-s)^{-\frac{1}{8}}\exp(\frac{1}{12}s^3),$$
$$u(s)=-4\Delta(s).$$

5 Remarks on the previous result

Tracy and Widom found good stuff in 1993, see [14], which is known as the GUE Tracy-Widom distribution. Its cumulative distribution function can be given as the Fredholm determinant

$$F_2(s) = \det(I - A_s).$$

where A_s is the operator on square integrable function on the half line (s, ∞) with kernel given in terms of Airy functions Ai by

$$\frac{A_i(x)A_i'(y) - A_i'(x)A_i(y)}{x - y}.$$

One alternative definition in terms of a solution to Painlevé II is

$$F_2(s) = \exp[-\int_s^\infty (x - s)q^2(s)ds].$$

where q is a solution of

$$q''(s) = sq(s) + 2q^3(s).$$

This function $F_2(s)$ is exactly what we wrote above. i.e. In our expression, $F_2(s)$ represents the large behaviour of the cumulative distribution function of Tracy-Widom distribution.

Then Choup used Edgeworth expansion to investigate the correction term for the problem of GUE&LUE in 2006. From his paper, see [11], we know the result below

The scaling is

$$t = 4(n + c_L) + 2\alpha + 2(2n)^{\frac{1}{3}}s. \quad (5.1)$$

His Edgeworth expansion is

$$F_{n,2}^{G,L}(t) = F_2(s)\{1 + a_2^{G,L}u_0(s)n^{-\frac{1}{3}} + b_2^{G,L}E_{c,2}(s)n^{-\frac{2}{3}}\} + O(n^{-1}). \quad (5.2)$$

where $a_2^L = 2^{\frac{2}{3}}c_L$. What he has done is valid for both Gaussian Unitary Ensemble and Laguerre Unitary Ensemble. Hence our problem can also be applied. Note that $F_2(s)$ above is also the cumulative distribution function of Tracy-Widom distribution. In the rest of the paper, Choup gave long calculations for u_0 and $E_{c,2}$. Finally he had not found an explicit formula for u_0, which is quite disappointing. Honestly, an implicit solution does not give us many real interests.

In our paper, we depart from a change of variable to construct a bridge between Laguerre weight and Jacobi time-dependent weight. Then we can derive a Painlevé differential equation by doing classical work in orthogonal polynomials. Then we try a further expansion to collect more information and at last make the appearance of the correction $u(s)$.

It is quite interesting to compare what we have done with Choup's work. Review $(*)$ in section 4.2 and (5.1), the two scalings are perfectly compatible with $c_L = -\frac{\alpha}{2}$. (4.15) and (5.2) also bring us a good news that the order of the correction term is correct.

6 Appendix

Note: The commands (words boxed) in this appendix should be computed by mathematica.

6.1 The σ-form of P_V in our problem

The scaling (∗) is defined as below

$$\boxed{t = 4 * n - c * s * n\hat{}(1/3);}$$

c0 is the constant where $\frac{d}{dt} = \text{c0} \times \frac{d}{ds}$,

$$\boxed{c0 = -1/c/(n\hat{}(1/3));}$$

Define some coefficients

$$\boxed{c1 = (2 * n - t + a);}$$

$$\boxed{d0 = t\hat{}2 * \text{c0}\hat{}4;}$$

$$\boxed{d1 = \text{c0} * \text{c1};}$$

$$\boxed{y = H[s];}$$

Now type the differential equation

$$\boxed{\text{eq:=}d0 * (D[y, \{s, 2\}])\hat{}2 ==}$$

$$\boxed{(y + d1 * D[y, s])\hat{}2 + 4 * (y - n * (n + a) - t * c0 * D[y, s]) * ((c0 * D[y, s])\hat{}2)}$$

Then we want expand the equation above by typing
$\boxed{\text{Expand[eq]}}$

$$\frac{16n^{2/3}H''[s]^2}{c^4} - \frac{8sH''[s]^2}{c^3} + \frac{s^2H''[s]^2}{c^2n^{2/3}} == H[s]^2 - \frac{2aH[s]H'[s]}{cn^{1/3}} + \frac{4n^{2/3}H[s]H'[s]}{c} - 2sH[s]H'[s] + \frac{a^2H'[s]^2}{c^2n^{2/3}} - \frac{8an^{1/3}H'[s]^2}{c^2} + \frac{2asH'[s]^2}{cn^{1/3}} - \frac{4n^{2/3}sH'[s]^2}{c} + s^2H'[s]^2 + \frac{4H[s]H'[s]^2}{c^2n^{2/3}} + \frac{16H'[s]^3}{c^3} - \frac{4sH'[s]^3}{c^2n^{2/3}}$$

6.2 Study on the leading-order

Let $\sigma_n(s) = \frac{H(s)}{c^2n^{\frac{2}{3}}}$, (4.5) in the text. In mathematica, we enter

$$\boxed{y = y * (c\hat{}2) * n\hat{}(2/3);}$$

Then we expand again the equation.

Expand[eq]

$16n^2H''[s]^2-8cn^{4/3}sH''[s]^2+c^2n^{2/3}s^2H''[s]^2 == c^4n^{4/3}H[s]^2-2ac^3nH[s]H'[s]+4c^3n^2H[s]H'[s]-2c^4n^{4/3}sH[s]H'[s]+a^2c^2n^{2/3}H'[s]^2-8ac^2n^{5/3}H'[s]^2+2ac^3nsH'[s]^2-4c^3n^2sH'[s]^2+c^4n^{4/3}s^2H'[s]^2+4c^4n^{4/3}H[s]H'[s]^2+16c^3n^2H'[s]^3-4c^4n^{4/3}sH'[s]^3$

Rewrite the result in the form of f(H,s)=0. We put all the terms on the left hand side and obtain

f=16 $n^2H''[s]^2 - 8cn^{4/3}sH''[s]^2 + c^2n^{2/3}s^2H''[s]^2 - (c^4n^{4/3}H[s]^2 - 2ac^3nH[s]H'[s] + 4c^3n^2H[s]H'[s] - 2c^4n^{4/3}sH[s]H'[s] + a^2c^2n^{2/3}H'[s]^2 - 8ac^2n^{5/3}H'[s]^2 + 2ac^3nsH'[s]^2 - 4c^3n^2sH'[s]^2 + c^4n^{4/3}s^2H'[s]^2 + 4c^4n^{4/3}H[s]H'[s]^2 + 16c^3n^2H'[s]^3 - 4c^4n^{4/3}sH'[s]^3)$;

Coefficient[f, n^2]

$-4c^3H[s]H'[s] + 4c^3sH'[s]^2 - 16c^3H'[s]^3 + 16H''[s]^2$

This is the equation on the leading-order.

$$-4c^3H[s]H'[s] + 4c^3sH'[s]^2 - 16c^3H'[s]^3 + 16H''[s]^2 = 0$$

6.3 Futher expansion

Now try $p + p_1n^\gamma = \sigma_n = \frac{H}{c^2n^{\frac{2}{3}}}$

$$y = (p[s] + p1[s] * (n\hat{}\gamma)) * (c\hat{}2 * n\hat{}(2/3));$$

Expand[eq];

We have explained why γ must be $\frac{1}{3}$ in section 4.3 then we study directly in this case. One can remove the semicolon in the command above to see the full result.

$\gamma = -\frac{1}{3}$;

Expand[eq]

$16n^2p''[s]^2-8cn^{4/3}sp''[s]^2+c^2n^{2/3}s^2p''[s]^2+32n^{5/3}p''[s]\mathrm{p1}''[s]-16cnsp''[s]\mathrm{p1}''[s]+2c^2n^{1/3}s^2p''[s]\mathrm{p1}''[s]+16n^{4/3}\mathrm{p1}''[s]^2-8cn^{2/3}s\mathrm{p1}''[s]^2+c^2s^2\mathrm{p1}''[s]^2 == c^4n^{4/3}p[s]^2+2c^4np[s]\mathrm{p1}[s]+c^4n^{2/3}\mathrm{p1}[s]^2-2ac^3np[s]p'[s]+4c^3n^2p[s]p'[s]-2c^4n^{4/3}sp[s]p'[s]-2ac^3n^{2/3}\mathrm{p1}[s]p'[s]+4c^3n^{5/3}\mathrm{p1}[s]p'[s]-2c^4ns\mathrm{p1}[s]p'[s]+a^2c^2n^{2/3}p'[s]^2-8ac^2n^{5/3}p'[s]^2+2ac^3nsp'[s]^2-4c^3n^2sp'[s]^2+c^4n^{4/3}s^2p'[s]^2+4c^4n^{4/3}p[s]p'[s]^2+4c^4n\mathrm{p1}[s]p'[s]^2+16c^3n^2p'[s]^3-4c^4n^{4/3}sp'[s]^3-2ac^3n^{2/3}p[s]\mathrm{p1}'[s]+4c^3n^{5/3}p[s]\mathrm{p1}'[s]-2c^4nsp[s]\mathrm{p1}'[s]-2ac^3n^{1/3}\mathrm{p1}[s]\mathrm{p1}'[s]+4c^3n^{4/3}\mathrm{p1}[s]\mathrm{p1}'[s]-2c^4n^{2/3}s\mathrm{p1}[s]\mathrm{p1}'[s]+2a^2c^2n^{1/3}p'[s]\mathrm{p1}'[s]-16ac^2n^{4/3}p'[s]\mathrm{p1}'[s]+4ac^3n^{2/3}sp'[s]\mathrm{p1}'[s]-8c^3n^{5/3}sp'[s]\mathrm{p1}'[s]+2c^4ns^2p'[s]\mathrm{p1}'[s]+$

$8c^4np[s]p'[s]\mathrm{p1}'[s]{+}8c^4n^{2/3}\mathrm{p1}[s]p'[s]\mathrm{p1}'[s]{+}48c^3n^{5/3}p'[s]^2\mathrm{p1}'[s]{-}12c^4nsp'[s]^2\mathrm{p1}'[s]{+}$ $a^2c^2\mathrm{p1}'[s]^2{-}8ac^2n\mathrm{p1}'[s]^2{+}2ac^3n^{1/3}s\mathrm{p1}'[s]^2{-}4c^3n^{4/3}s\mathrm{p1}'[s]^2{+}c^4n^{2/3}s^2\mathrm{p1}'[s]^2{+}$ $4c^4n^{2/3}p[s]\mathrm{p1}'[s]^2{+}4c^4n^{1/3}\mathrm{p1}[s]\mathrm{p1}'[s]^2{+}48c^3n^{4/3}p'[s]\mathrm{p1}'[s]^2{-}12c^4n^{2/3}sp'[s]\mathrm{p1}'[s]^2{+}$ $16c^3n\mathrm{p1}'[s]^3 - 4c^4n^{1/3}s\mathrm{p1}'[s]^3$

With the same idea, we rearrange the above equation as $g(p, p1, s) = 0$. The left-hand-side is

g=16 $\mathrm{n}^2p''[s]^2 - 8cn^{4/3}sp''[s]^2 + c^2n^{2/3}s^2p''[s]^2 + 32n^{5/3}p''[s]p1''[s] -$ $16cnsp''[s]p1''[s] + 2c^2n^{1/3}s^2p''[s]p1''[s] + 16n^{4/3}p1''[s]^2 -$ $8cn^{2/3}sp1''[s]^2 + c^2s^2p1''[s]^2 -$ $(c^4n^{4/3}p[s]^2 + 2c^4np[s]p1[s] + c^4n^{2/3}\mathrm{p1}[s]^2 - 2ac^3np[s]p'[s] +$ $4c^3n^2p[s]p'[s] - 2c^4n^{4/3}sp[s]p'[s] - 2ac^3n^{2/3}\mathrm{p1}[s]p'[s] +$ $4c^3n^{5/3}p1[s]p'[s] - 2c^4nsp1[s]p'[s] + a^2c^2n^{2/3}p'[s]^2 - 8ac^2n^{5/3}p'[s]^2 +$ $2ac^3nsp'[s]^2 - 4c^3n^2sp'[s]^2 + c^4n^{4/3}s^2p'[s]^2 + 4c^4n^{4/3}p[s]p'[s]^2 +$ $4c^4np1[s]p'[s]^2 + 16c^3n^2p'[s]^3 - 4c^4n^{4/3}sp'[s]^3 - 2ac^3n^{2/3}p[s]p1'[s] +$ $4c^3n^{5/3}p[s]p1'[s] - 2c^4nsp[s]p1'[s] - 2ac^3n^{1/3}\mathrm{p1}[s]p1'[s] +$ $4c^3n^{4/3}p1[s]p1'[s] - 2c^4n^{2/3}sp1[s]p1'[s] + 2a^2c^2n^{1/3}p'[s]p1'[s] -$ $16ac^2n^{4/3}p'[s]p1'[s] + 4ac^3n^{2/3}sp'[s]p1'[s] - 8c^3n^{5/3}sp'[s]p1'[s] +$ $2c^4ns^2p'[s]p1'[s] + 8c^4np[s]p'[s]p1'[s] + 8c^4n^{2/3}\mathrm{p1}[s]p'[s]p1'[s] +$ $48c^3n^{5/3}p'[s]^2p1'[s] - 12c^4nsp'[s]^2p1'[s] + a^2c^2\mathrm{p1}'[s]^2 -$ $8ac^2np1'[s]^2 + 2ac^3n^{1/3}sp1'[s]^2 - 4c^3n^{4/3}sp1'[s]^2 + c^4n^{2/3}s^2p1'[s]^2 +$ $4c^4n^{2/3}p[s]p1'[s]^2 + 4c^4n^{1/3}\mathrm{p1}[s]p1'[s]^2 + 48c^3n^{4/3}p'[s]p1'[s]^2 -$ $12c^4n^{2/3}sp'[s]p1'[s]^2 + 16c^3np1'[s]^3 - 4c^4n^{1/3}sp1'[s]^3$;

Coefficient$[g, n\hat{}2]$

$-4c^3p[s]p'[s] + 4c^3sp'[s]^2 - 16c^3p'[s]^3 + 16p''[s]^2$
which is exactly the equation on the leading-order as we have seen previously, namely (4.5) in the paper.

$$-4c^3p[s]p'[s] + 4c^3sp'[s]^2 - 16c^3p'[s]^3 + 16p''[s]^2 = 0$$

Coefficient$[g, n\hat{}(5/3)]$

$-4c^3\mathrm{p1}[s]p'[s] + 8ac^2p'[s]^2 - 4c^3p[s]\mathrm{p1}'[s] + 8c^3sp'[s]\mathrm{p1}'[s] - 48c^3p'[s]^2\mathrm{p1}'[s] +$ $32p''[s]\mathrm{p1}''[s]$
which is the equation on the second-leading-order, namely (4.6) in the paper.

$$\begin{aligned} &- 4c^3p1[s]p'[s] + 8ac^2p'[s]^2 - 4c^3p[s]p1'[s] \\ &+ 8c^3sp'[s]p1'[s] - 48c^3p'[s]^2p1'[s] + 32p''[s]p1''[s] = 0 \end{aligned}$$

This equation allows us to study the asymptotic behaviour of the correction term for large n.

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